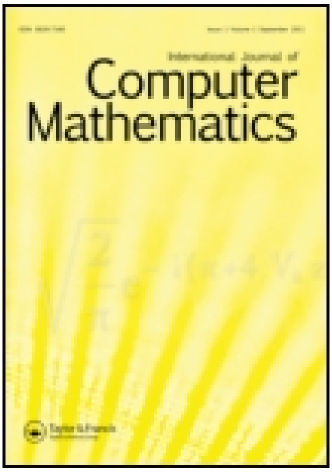

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Two-level quadratic equal-order stabilized method for the Stokes eigenvalue problem

Zhifeng Weng[a], Jerry Zhijian Yang[a] & Xiliang Lu[a]
[a] School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China

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Two-level quadratic equal-order stabilized method for the Stokes eigenvalue problem

Zhifeng Weng, Jerry Zhijian Yang and Xiliang Lu*

School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China

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In this article, the $P_2 - P_2$-stabilized finite element method based on two local Gaussian quadratures is applied to discretize the Stokes eigenvalue problem, and the corresponding convergence analysis is given. Furthermore, a two-level scheme, which solves the Stokes eigenvalue problem on a coarser grid and a Stokes problem on the fine grid, is employed to reduce the computational cost. Numerical examples are given to confirm the theoretical results.

Keywords: two-level method; Stokes eigenvalue problem; quadratic equal-order stabilized method

2000 AMS Subject Classifications: 65M60; 76D07; 65M12

1. Introduction

The eigenvalue problems play an important role in many areas such as structural engineering and fluid mechanics [2]. Meanwhile, numerical approximation of eigenvalue problems has received increasing attention in the past decades [6,21,27]. Lovadina *et al.* [21] proposed a posteriori error estimates for the Stokes eigenvalue problem. Yin *et al.* [27] derived a general procedure by an asymptotic expansion for the eigenvalues of the Stokes problem on a rectangular mesh. Chen and Lin [6] proposed the stream function–vorticity–pressure method for the eigenvalue problem.

For the Stokes problem, the mixed finite element method is frequently used to approximate the velocity and the pressure simultaneously. The two discretization spaces for the velocity and the pressure must be chosen carefully such that the discrete inf–sup condition is satisfied to ensure the stability. However, in practice, some mixed finite element pairs which do not satisfy the discrete inf–sup condition are also useful due to their simple formula (e.g. $P_1 - P_1$) or the higher order accuracy (e.g. $P_2 - P_2$). The equal-order finite element pairs are of practical importance in scientific computation and are computationally convenient in a parallel processing and multigrid context. For those elements, extra stabilization terms must be added to the weak formulation. Several authors have recently proposed the stabilization of the low equal-order finite element by projection-based stabilization techniques, which include pressure-gradient projection (PGP) stabilized method [3,4,8] and pressure projection stabilized method [5,9,20]. These stabilization techniques do not require specification of stabilizing parameters or edge-based data structures.

*Corresponding author. Email: xllv.math@whu.edu.cn

In the first part of this paper, we introduce the quadratic equal-order finite element pair ($P_2 - P_2$) based on the PGP stabilized method to discretize the Stokes eigenvalue problem, which aims to extend the work of Bochev *et al.* [5], Li *et al.* [20], Huang *et al.* [14] and Zheng *et al.* [28] for Stokes equations. The method offsets the discrete pressure gradient space to circumvent the inf–sup condition. We project the pressure-gradient space onto the piecewise constant function space and use two local Gaussian quadratures to deal with the stabilization term.

Recently, many methods have been proposed and analysed to reduce the computational cost for the approximations of the eigenvalue problem by the finite element methods. The two-level discretization method is one of the efficient methods and has been well developed. It was first introduced by Xu [23,24] for nonsymmetric and nonlinear elliptic problems. Then the technique has been successfully applied in the studies of Xu *et al.* [25], Layton *et al.* [17,18], He *et al.* [12,13] and Yang *et al.* [26]. In particular, Huang *et al.* [15] proposed a two-level discretization scheme for solving Stokes eigenvalue problem by the stabilization of the lowest equal-order finite element pair $P_1 - P_1$ using the projection of the pressure onto the piecewise constant space. Based on the above studies, the second part of this paper is to propose and analyse a two-level algorithm which can reduce the computational cost of the eigenpair approximations for the Stokes eigenvalue problem by a $P_2 - P_2$-stabilized finite element pair based on the PGP method.

The rest of the paper is organized as follows. In the next section, we introduce the notation and some well-known results which will be used throughout the article. The PGP method for Stokes eigenvalue problem is given in Section 3. In Section 4, a two-level algorithm and its error estimates are provided. In Section 5, numerical results are presented to confirm the theoretical analysis and the efficiency of the proposed method. Finally, we will conclude our presentation in Section 6 with a few comments and topics for further research.

2. Preliminaries

We shall use the following Hilbert spaces:

$$\mathbf{V} = H_0^1(\Omega)^2, \quad \mathbf{Y} = L^2(\Omega)^2, \quad W = L_0^2(\Omega) = \left\{ q \in L^2(\Omega), \int_\Omega q \, \mathrm{d}x = 0 \right\}.$$

The standard Sobolev spaces $L^2(\Omega)^m$, where $m = 1, 2$, are equipped with the L^2-scalar product $(\cdot, \cdot)$ and L^2-norm $\| \cdot \|_0$. The norm and seminorm in $H^k(\Omega)^m$ are denoted by $\| \cdot \|_k$ and $| \cdot |_k$, respectively. The space $\mathbf{V}$ is equipped with the norm $\|\nabla \cdot \|_0$ or its equivalent norm $\| \cdot \|_1$ due to Poincaré inequality. Spaces consisting of vector-valued functions are denoted in boldface.

The Stokes eigenvalue problem reads as follows:

Find $(\mathbf{u}, p, \lambda) \in \mathbf{V} \times W \times \mathbb{R}$ such that $\|\mathbf{u}\|_0 = 1$ and

$$-\nu \Delta \mathbf{u} + \nabla p = \lambda \mathbf{u} \quad \text{in } \Omega, \tag{1}$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \tag{2}$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega, \tag{3}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with C^1 boundary $\partial\Omega$. $\lambda \in \mathbb{R}$ is the eigenvalue and $\mathbf{u}$ and p are the velocity and pressure, respectively. $\mathbb{R}$ is the set of all real numbers. Here, ν is the kinematic viscosity and we set $\nu = 1$ for simplicity.

We define the continuous bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and $r(\cdot,\cdot)$ on $\mathbf{V}\times\mathbf{V}$, $\mathbf{V}\times W$ and $\mathbf{Y}\times\mathbf{Y}$ as follows:

$$a(\mathbf{u},\mathbf{v}) = (\nabla\mathbf{u},\nabla\mathbf{v}) \quad \forall\mathbf{u},\mathbf{v}\in\mathbf{V},$$
$$b(\mathbf{v},q) = (\operatorname{div}\mathbf{v},q) \quad \forall\mathbf{v}\in\mathbf{V},\ \forall q\in W,$$
$$r(\mathbf{u},\mathbf{v}) = (\mathbf{u},\mathbf{v}) \quad \forall\mathbf{v},\mathbf{u}\in\mathbf{Y}.$$

Let us also define the generalized bilinear form $B((\cdot,\cdot),(\cdot,\cdot))$ on $(\mathbf{V}\times W)\times(\mathbf{V}\times W)$ as follows:

$$B((\mathbf{u},p),(\mathbf{v},q)) = a(\mathbf{u},\mathbf{v}) - b(\mathbf{v},p) - b(\mathbf{u},q) \quad \forall(\mathbf{u},p),(\mathbf{v},q)\in\mathbf{V}\times W. \tag{4}$$

By the above notation, the variational form associating with systems (1)–(3) is as follows: find $(\mathbf{u},p;\lambda)\in(\mathbf{V}\times W)\times\mathbb{R}$ with $\|\mathbf{u}\|_0 = 1$, such that

$$B((\mathbf{u},p),(\mathbf{v},q)) = \lambda r(\mathbf{u},\mathbf{v}), \quad \forall(\mathbf{v},q)\in\mathbf{V}\times W. \tag{5}$$

It has been shown in [2] that Equation (5) admits a countable set of real eigenvalues

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots,$$

and the corresponding eigenfunctions

$$(\mathbf{u}_1,p_1),(\mathbf{u}_2,p_2),(\mathbf{u}_3,p_3),\ldots,$$

with $r(\mathbf{u}_i,\mathbf{u}_j) = \delta_{ij}$.

Let the $M(\lambda_i)$ be the eigenspace associated with λ_i, i.e.

$$M(\lambda_i) = \{\mathbf{u}\in\mathbf{V}, \mathbf{u} \text{ is an eigenfunction of (5) corresponding to } \lambda_i\}.$$

Moreover, the bilinear form $b(\cdot,\cdot)$ satisfies the inf–sup condition:

$$\sup_{\mathbf{v}\in\mathbf{V}} \frac{|b(\mathbf{v},q)|}{\|\mathbf{v}\|_1} \ge \beta\|q\|_0 \quad \forall q\in W, \tag{6}$$

where $\beta > 0$ is a constant. Therefore, the generalized bilinear form B satisfies the continuity property and coercivity property

$$|B((\mathbf{u},p),(\mathbf{v},q))| \le C(\|\mathbf{u}\|_1 + \|p\|_0)\times(\|\mathbf{v}\|_1 + \|q\|_0), \tag{7}$$

$$\sup_{(\mathbf{v},q)\in(\mathbf{V},W)} \frac{|B((\mathbf{u},p),(\mathbf{v},q))|}{\|\mathbf{v}\|_1 + \|q\|_0} \ge \beta_1(\|\mathbf{u}\|_1 + \|p\|_0), \tag{8}$$

where $C > 0$ and $\beta_1 > 0$ are the constants depending only on Ω. Throughout the paper, we use c or C to denote a generic positive constant the value of which may change from place to place but independent of its mesh size.

3. A quadratic equal-order stabilized method

The finite element subspaces $(\mathbf{V}_h, W_h)\subset(\mathbf{V},W)$, which are associated with a regular triangulation T_h of Ω in [7], are chosen by the $P_2 - P_2$ pair as follows:

$$\mathbf{V}_h = \{v_h = (v_1,v_2)\in(C^0(\Omega))^2\cap\mathbf{V} : v_i|_T\in P_2(T)\ \ \forall T\in T_h,\ i=1,2\}, \tag{9}$$

$$W_h = \{w\in C^0\cap W : w|_T\in P_2(T)\ \ \forall T\in T_h\}, \tag{10}$$

where $P_2(T)$ represents the space of quadratic polynomial functions on T.

It is apparent that this choice of the approximate spaces $\mathbf{V}_h$ and W_h does not satisfy the inf–sup condition [28]. Then, we give a stabilized finite-element approximation based on the PGP stabilization method (see [5,28]). The idea is as follows:

Let $\Pi : L^2(\Omega) \to R_0$ be the standard L^2-projection:

$$(p, q) = (\Pi p, q) \quad \forall p \in W, q \in R_0, \tag{11}$$

where $R_0 = \{q \in W : q|_T \in P_0(T), \forall T \in T_h\}$.

The projection operator Π has the following properties [7]:

$$\|\Pi p\|_0 \leq c\|p\|_0 \qquad \forall p \in W, \tag{12}$$

$$\|p - \Pi p\|_0 \leq ch\|p\|_1 \quad \forall p \in H^1(\Omega). \tag{13}$$

The PGP stabilization term is given by

$$G(p, q) = (\nabla p - \Pi\nabla p, \nabla q - \Pi\nabla q) \quad \forall p, q \in W_h. \tag{14}$$

By adding the stabilization term into the generalized bilinear form $B((\cdot,\cdot),(\cdot,\cdot))$, we define

$$B_h((\mathbf{u}_h, p_h), (\mathbf{v}, q)) = B((\mathbf{u}_h, p_h), (\mathbf{v}, q)) - G(p_h, q) \quad \forall(\mathbf{u}_h, p_h), (\mathbf{v}, q) \in \mathbf{V}_h \times W_h. \tag{15}$$

Then the corresponding discrete variational formulation for the Stokes eigenvalue problem reads: find $(\mathbf{u}_h, p_h; \lambda_h) \in (\mathbf{V}_h \times W_h) \times \mathbb{R}$ with $\|\mathbf{u}_h\|_0 = 1$, such that

$$B_h((\mathbf{u}_h, p_h), (\mathbf{v}, q)) = \lambda_h r(\mathbf{u}_h, \mathbf{v}) \quad \forall(\mathbf{v}, q) \in \mathbf{V}_h \times W_h. \tag{16}$$

By choosing $\mathbf{v} = \mathbf{u}_h$ and $q = -p_h$, and using the fact that the bilinear form $a(\cdot,\cdot)$ is positive-definite, we find that the discrete eigenvalues λ_{jh} are positive. Let the eigenvalue of Equation (16) be

$$0 < \lambda_{1h} \leq \lambda_{2h} \leq \lambda_{3h} \leq \cdots \leq \lambda_{Nh},$$

and the corresponding eigenfunctions

$$(\mathbf{u}_{1h}, p_{1h}), (\mathbf{u}_{2h}, p_{2h}), (\mathbf{u}_{3h}, p_{3h}), \ldots, (\mathbf{u}_{Nh}, p_{Nh}),$$

where $r(\mathbf{u}_{ih}, \mathbf{u}_{jh}) = \delta_{ij}$, $1 \leq i, j \leq N$, N denotes the dimension of the finite element space.

Similarly, let $M_h(\lambda_{ih})$ be the eigenspace associated with λ_{ih}:

$$M_h(\lambda_{ih}) = \{\mathbf{u}_h \in \mathbf{V}_h, \mathbf{u}_h \text{ is an eigenfunction of Equation (16) corresponding to } \lambda_{ih}\}.$$

The next theorem shows the continuity property and the weak coercivity property of the bilinear form $B_h((\mathbf{u}_h, p_h), (\mathbf{v}, q))$ for the finite element space $\mathbf{V}_h \times W_h$.

THEOREM 3.1 *For all* $(\mathbf{u}_h, p_h), (\mathbf{v}, q) \in \mathbf{V}_h \times W_h$, *there exist positive constants* C *and* β_2, *independent of* h, *such that*

$$|B_h((\mathbf{u}_h, p_h), (\mathbf{v}, q))| \leq C(\|\mathbf{u}_h\|_1 + \|p_h\|_0) \times (\|\mathbf{v}\|_1 + \|q\|_0), \tag{17}$$

$$\sup_{(\mathbf{v},q)\in(\mathbf{V}_h,W_h)} \frac{|B_h((\mathbf{u}_h, p_h), (\mathbf{v}, q))|}{\|\mathbf{v}\|_1 + \|q\|_0} \geq \beta_2(\|\mathbf{u}_h\|_1 + \|p_h\|_0). \tag{18}$$

Proof Continuity of B_h can be verified directly (cf. [28, p. 1184]). The inf–sup condition (18) is Theorem 4.2 in [28].

Since the convergence of the finite element approximation to the eigenvalue problem depends on the regularity of the original eigenvalue problem, we first provide a regularity result of the eigenfunction

$$(\mathbf{u}, p) \in H^3(\Omega)^2 \times H^2(\Omega). \tag{19}$$

This can be verified by a bootstrap strategy, i.e. we fist note that the eigenfunction $\mathbf{u} \in \mathbf{V}$, then apply [16, Chapter 3, Theorem 2] to obtain the desired regularity.

To prove the convergence rate for the eigenpairs, we first consider the error estimations for the Stokes problem. To this end, we introduce the bounded operators $(G, S) : \mathbf{Y} \to (\mathbf{V}, W)$ and $(G_h, S_h) : \mathbf{Y} \to (\mathbf{V}_h, W_h)$ which are the solution to Stokes problem and its corresponding discretization problem, i.e. given any $\mathbf{f} \in \mathbf{Y}$, $(G\mathbf{f}, S\mathbf{f})$ satisfies

$$B((G\mathbf{f}, S\mathbf{f}), (\mathbf{v}, q)) = r(\mathbf{f}, \mathbf{v}) \quad \forall (\mathbf{v}, q) \in \mathbf{V} \times W, \tag{20}$$

and $(G_h\mathbf{f}, S_h\mathbf{f})$ solves

$$B_h((G_h\mathbf{f}, S_h\mathbf{f}), (\mathbf{v}, q)) = r(\mathbf{f}, \mathbf{v}) \quad \forall (\mathbf{v}, q) \in \mathbf{V}_h \times W_h. \tag{21}$$

■

LEMMA 3.1 *For any* $\mathbf{f} \in \mathbf{H}^1(\Omega)^2$, *the following error estimate holds:*

$$\|G\mathbf{f} - G_h\mathbf{f}\|_0 + h(\|G\mathbf{f} - G_h\mathbf{f}\|_1 + \|S\mathbf{f} - S_h\mathbf{f}\|_0) \le ch^3\|\mathbf{f}\|_1. \tag{22}$$

Proof By [16, Chapter 3, Theorem 2], we have the following a priori estimate for Stokes problem:

$$\|G\mathbf{f}\|_3 + \|S\mathbf{f}\|_2 \le c\|\mathbf{f}\|_1.$$

Then by Theorem 4.3 of [28], a priori error estimate for the Stokes equation problem based on the stabilized mixed finite element method is given as: for all $\mathbf{f} \in H^1(\Omega)^2$, we have

$$\|G\mathbf{f} - G_h\mathbf{f}\|_1 + \|S\mathbf{f} - S_h\mathbf{f}\|_0 \le ch^2\|\mathbf{f}\|_1. \tag{23}$$

To obtain the L^2-estimation for $\mathbf{u}$, we use a standard duality argument. For the finite-element spaces $\mathbf{V}_h$ and W_h, it is known that the standard interpolation operator $I_h : H^2(\Omega) \cap W \to W_h$ and $J_h : \mathbf{H}^3(\Omega) \cap \mathbf{V} \to \mathbf{V}_h$ satisfy the following inequalities (cf. [10, p. 217]):

$$\|\mathbf{u} - J_h\mathbf{u}\|_0 + h(\|\mathbf{u} - J_h\mathbf{u}\|_1 + \|p - I_hp\|_0) \le ch^3(\|p\|_2 + \|\mathbf{u}\|_3). \tag{24}$$

The duality argument is similar to the one in [19], i.e. define the dual problem: find $(\psi_1, \phi_1) \in (\mathbf{V}, W)$ such that

$$B((\psi_1, \phi_1); e, \eta) = (e, G\mathbf{f} - G_h\mathbf{f}), \quad \forall (e, \eta) \in (\mathbf{V}, W).$$

Then

$$\|\psi_1\|_3 + \|\phi_1\|_2 \le \|G\mathbf{f} - G_h\mathbf{f}\|_1. \tag{25}$$

From Equations (20) and (21), we have

$$B_h((\psi_h, \phi_h); (G\mathbf{f} - G_h\mathbf{f}, S\mathbf{f} - S_h\mathbf{f})) = 0 \quad \forall (\psi_h, \phi_h) \in (\mathbf{V}_h, W_h).$$

Setting $e = G\mathbf{f} - G_h\mathbf{f}, \eta = S\mathbf{f} - S_h\mathbf{f}$ and $(\psi_h, \phi_h) = (J_h\psi, I_h\phi) \in (\mathbf{V}_h, W_h)$, then using Equations (17), (24), (25) and (23), we can obtain

$$\begin{aligned}\|G\mathbf{f} - G_h\mathbf{u}\|_0^2 &= B_h((\psi_1 - \psi_h, \phi_1 - \phi_h); (G\mathbf{f} - G_h\mathbf{f}, S\mathbf{f} - S_h\mathbf{f}))\\ &\le C(\|\psi_1 - \psi_h\|_1 + \|\phi_1 - \phi_h\|_0)(\|G\mathbf{f} - G_h\mathbf{f}\|_1 + \|S\mathbf{f} - S_h\mathbf{f}\|_0)\\ &\le Ch^2(\|G\mathbf{f} - G_h\mathbf{f}\|_1 + \|S\mathbf{f} - S_h\mathbf{f}\|_0)(\|\psi_1\|_3 + \|\phi_1\|_2)\\ &\le Ch^2(\|G\mathbf{f} - G_h\mathbf{f}\|_1 + \|S\mathbf{f} - S_h\mathbf{f}\|_0)\|G\mathbf{f} - G_h\mathbf{f}\|_1 \le Ch^6\|\mathbf{f}\|_1^2, \qquad (26)\end{aligned}$$

which completes the proof. ■

The next theorem gives the convergence result of eigenfunctions and eigenvalues for the Stokes eigenvalue problem.

THEOREM 3.2 *Let* $(\mathbf{u}_h, p_h; \lambda_h)$ *be the ith discrete eigenpair of* (16). *Then there exists an ith eigenpair* $(\mathbf{u}, p; \lambda)$ *of* (5) *which satisfies the following error estimates:*

$$\|\mathbf{u} - \mathbf{u}_h\|_0 + h(\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0) \le ch^3, \qquad (27)$$

$$|\lambda - \lambda_h| \le ch^4. \qquad (28)$$

Proof By Lemma 3.1, error for the operator norm

$$\|G - G_h\|_1 \triangleq \sup_{\mathbf{f}\in\mathbf{H}^1(\Omega)^2} \frac{\|G\mathbf{f} - G_h\mathbf{f}\|_1}{\|\mathbf{f}\|_1}, \quad \|S - S_h\|_0 \triangleq \sup_{\mathbf{f}\in\mathbf{H}^1(\Omega)^2} \frac{\|S\mathbf{f} - S_h\mathbf{f}\|_0}{\|\mathbf{f}\|_1}$$

can be estimated by

$$\|G - G_h\|_1 + \|S - S_h\|_0 \le ch^2.$$

Following the discussion in [2, p. 699] or [25, Proposition 3.1], for the *i*th discrete eigenpair $(\mathbf{u}_h, p_h; \lambda_h)$, there exists an *i*th eigenpair $(\mathbf{u}, p; \lambda)$ such that

$$\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0 \le c(\|G - G_h\|_1 + \|S - S_h\|_0) \le ch^2,$$

$$\|\lambda - \lambda_h| \le c\|G - G_h\|_1^2 \le ch^4.$$

Moreover, by $\|G\mathbf{f} - G_h\mathbf{f}\|_0 \le ch^3\|\mathbf{f}\|_1$ in Lemma 3.1, the standard argument (cf. [22, p. 448]) leads

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \le ch^3,$$

which completes the proof. ■

Remark 3.1 Compared with the error estimates of the eigenvector and eigenvalue for the $P_1 - P_1$ element in [15]:

$$\|\mathbf{u} - \mathbf{u}_h\|_0 + h(\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0) \le ch^2, \qquad (29)$$

$$|\lambda - \lambda_h| \le ch^2, \qquad (30)$$

our $P_2 - P_2$-stabilized method predicts a convergence rate of order $O(h^2)$ for $\mathbf{u}$ in the H^1-norm and for p in the L^2-norm which is one order higher than $P_1 - P_1$-stabilized method. And the eigenvalue approximation is two orders higher than the $P_1 - P_1$-stabilized method.

4. Two-level stabilized finite element method

In this section, we shall present a two-level algorithm to reduce the computational cost.

Let H and $h \ll H$ be two real positive parameters corresponding to two different grid meshes. A coarse mesh triangulation $T_H(\Omega)$ of Ω is made as in Section 3 and a fine mesh triangulation $T_h(\Omega)$ is generated by a mesh refinement process to $T_H(\Omega)$. The conforming finite element space pairs $(\mathbf{V}_h, W_h)$ and $(\mathbf{V}_H, W_H) \subset (\mathbf{V}_h, W_h)$ based on the triangulations $T_h(\Omega)$ and $T_H(\Omega)$, respectively, are constructed as in Section 3. The two-level stabilized finite element approximation consists of three steps:

Step 1. On the coarse grid T_H, solve the following Stokes eigenvalue problem for $(p_H, \mathbf{u}_H; \lambda_H) \in (W_H \times \mathbf{V}_H) \times \mathbb{R}$ with $\|\mathbf{u}_H\|_0 = 1$:

$$B_H((\mathbf{u}_H, p_H), (\mathbf{v}, q)) = \lambda_H r(\mathbf{u}_H, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_H,\ q \in W_H. \tag{31}$$

Step 2. On the fine grid T_h, compute $(p_h, \mathbf{u}_h) \in W_h \times \mathbf{V}_h$ to satisfy the following Stokes problem:

$$B_h((\mathbf{u}_h, p_h), (\mathbf{v}, q)) = \lambda_H r(\mathbf{u}_H, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_h,\ q \in W_h. \tag{32}$$

Step 3. Compute the eigenvalue by the Rayleigh quotient

$$\lambda_h = \frac{B_h((\mathbf{u}_h, p_h), (\mathbf{u}_h, p_h))}{r(\mathbf{u}_h, \mathbf{u}_h)}. \tag{33}$$

Next, we will study the convergence of the two-level stabilized finite element solution. To do this, we define the Galerkin projection operator $(R_h, Q_h) : (\mathbf{V}, W) \to (\mathbf{V}_h, W_h)$ by

$$B_h((R_h(\mathbf{v}, q), Q_h(\mathbf{v}, q)), (\mathbf{v}_h, q_h)) = B((\mathbf{v}, q), (\mathbf{v}_h, q_h)) \quad \forall (\mathbf{v}_h, q_h) \in (\mathbf{V}_h, W_h). \tag{34}$$

By Theorem 3.1, (R_h, Q_h) is well defined and the following approximation properties are fulfilled.

LEMMA 4.1 *For all* $(\mathbf{u}, p) \in (\mathbf{H}^3(\Omega)^2 \cap \mathbf{V}, H^2(\Omega) \cap W)$, *we have*

$$\|\mathbf{u} - R_h(\mathbf{u}, p)\|_1 + \|p - Q_h(\mathbf{u}, p)\|_0 \le ch^2(\|\mathbf{u}\|_3 + \|p\|_2). \tag{35}$$

Proof Recall the interpolation operators J_h and I_h as in (24). Then by definition of Galerkin projection (34) and Theorem 3.1, we have

$$\begin{aligned}
&\|J_h\mathbf{u} - R_h(\mathbf{u}, p)\|_1 + \|I_h p - Q_h(\mathbf{u}, p)\|_0 \\
&\quad \le \beta_2^{-1} \sup_{(\mathbf{u}_h, p_h) \in (\mathbf{V}_h, W_h)} \frac{|B_h((J_h\mathbf{u} - R_h(\mathbf{u}, p), I_h p - Q_h(\mathbf{u}, p)), (\mathbf{u}_h, p_h))|}{\|\mathbf{u}_h\|_1 + \|p_h\|_0} \\
&\quad = \beta_2^{-1} \sup_{(\mathbf{u}_h, p_h) \in (\mathbf{V}_h, W_h)} \frac{|B_h((J_h\mathbf{u} - u, I_h p - p), (\mathbf{u}_h, p_h))|}{\|\mathbf{u}_h\|_1 + \|p_h\|_0} \\
&\quad \le c(\|J_h\mathbf{u} - \mathbf{u}\|_1 + \|I_h p - p\|) \le ch^2(\|\mathbf{u}\|_3 + \|p\|_2).
\end{aligned}$$

Thus the triangle inequality implies the conclusion. ■

The following identity that relates the errors in the eigenvalue and eigenvector can be found in [1, Lemma 3.1].

LEMMA 4.2 *Let $(\mathbf{u}, p; \lambda)$ be an eigen-pair of (5), for any $\mathbf{s} \in \mathbf{V}\backslash\{0\}$ and $w \in W$, we have*

$$\frac{B((\mathbf{s}, w), (\mathbf{s}, w))}{r(\mathbf{s}, \mathbf{s})} - \lambda = \frac{B((\mathbf{s} - \mathbf{u}, w - p), (\mathbf{s} - \mathbf{u}, w - p))}{r(\mathbf{s}, \mathbf{s})} - \lambda \frac{r(\mathbf{s} - \mathbf{u}, \mathbf{s} - \mathbf{u})}{r(\mathbf{s}, \mathbf{s})}. \tag{36}$$

The next theorem gives the error estimates for our two-level scheme.

THEOREM 4.1 *Let $(\mathbf{u}_h, p_h; \lambda_h)$ be the ith discrete eigenpair. Then there exists an eigenpair $(\mathbf{u}, p; \lambda)$ of Stokes operator such that*

$$\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0 \le c(h^2 + H^3), \tag{37}$$

$$|\lambda - \lambda_h| \le c(h^4 + H^6). \tag{38}$$

Proof Denoting by $(e_h, \eta_h) = (R_h(\mathbf{u}, p) - \mathbf{u}_h, Q_h(\mathbf{u}, p) - p_h)$, we derive from Equations (5), (32) and (34),

$$B_h((e_h, \eta_h), (\mathbf{v}, q)) = \lambda r(\mathbf{u} - \mathbf{u}_H, \mathbf{v}) + (\lambda - \lambda_H) r(\mathbf{u}_H, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_h,\ q \in W_h. \tag{39}$$

Let $(v, q) = (e_h, \eta_h)$ in Equation (39), by using Theorem 3.1, Sobolev embedding theorem and Theorems 3.2, we obtain

$$\|e_h\|_1 + \|\eta_h\|_0 \le C(|\lambda - \lambda_H| \|\mathbf{u}_H\|_0 + \lambda \|\mathbf{u} - \mathbf{u}_H\|_0) \le cH^3. \tag{40}$$

Combining with Lemma 4.1, hence

$$\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0 \le c(H^3 + h^2). \tag{41}$$

Moreover, using Equation (33) and Lemma 4.2, we can obtain

$$\begin{aligned} \frac{B_h((\mathbf{u}_h, p_h), (\mathbf{u}_h, p_h))}{r(\mathbf{u}_h, \mathbf{u}_h)} - \lambda &= \frac{B((\mathbf{u}_h - \mathbf{u}, p_h - p), (\mathbf{u}_h - \mathbf{u}, p_h - p)) - G(p_h, p_h)}{r(\mathbf{u}_h, \mathbf{u}_h)} \\ &\quad - \lambda \frac{(\mathbf{u}_h - \mathbf{u}, \mathbf{u}_h - \mathbf{u})}{r(\mathbf{u}_h, \mathbf{u}_h)}. \end{aligned} \tag{42}$$

By virtue of Equation (7), we have

$$\begin{aligned} |\lambda - \lambda_h\| &\le C\|\mathbf{u} - \mathbf{u}_h\|_1^2 + C\|p - p_h\|_0^2 \\ &\le c(h^2 + H^3)^2. \end{aligned} \tag{43}$$

■

Remark 4.1 If we choose H and h such that $h = H^{3/2}$ for Theorem 4.1, then we obtain the convergence rate of the same order as the usual stabilized finite element method from Theorem 3.1. However, the two-level method is more efficient than the one-level scheme.

5. Numerical results

In this section we present numerical examples to check the numerical theory developed in the previous sections and illustrate the efficiency of our method based on the two local Gaussian quadratures. Our method is based on piecewise quadratic polynomial functions for both the

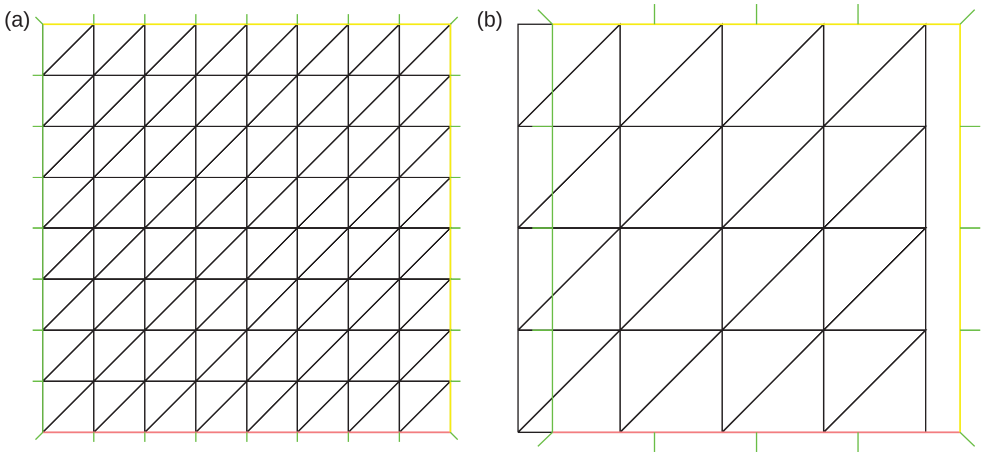

Figure 1. (a) Coarse grid division at $H = \frac{1}{4}$. (b) Fine grid division at $h = \frac{1}{8}$.

velocity and the pressure fields. The stabilized term which is defined by local Gaussian quadratures can be rewritten as

$$G(p,q) = \sum_{T \in T_h} \left(\int_{T,2} \nabla p \cdot \nabla q \, \mathrm{d}\mathbf{x} - \int_{T,1} \nabla p \cdot \nabla q \, \mathrm{d}\mathbf{x} \right) \quad \forall p, q \in W_h,$$

where $\int_{T,i} g(x,y)\,\mathrm{d}\mathbf{x}$ denotes a Gaussian quadrature over T which is exact for polynomials of degree i, where $i = 1, 2$.

In our numerical experiments, Ω is the unit square domain $[0,1] \times [0,1]$ in R^2. The domain Ω is uniformly divided by the triangulations of mesh size H and $h = H^{3/2}$, see, e.g. Figure 1. We denote by $\mathbf{U}$ the array of the velocity and by P the array of the pressure. It is easy to see that Equation (16) can be written in a matrix form as follows:

$$\begin{bmatrix} A & -B \\ -B^{\mathrm{T}} & -G \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ P \end{bmatrix} = \lambda_h \begin{bmatrix} E & O \\ O & O \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ P \end{bmatrix},$$

where the matrices A, B and E are deduced in $(a_{i,j})_{s\times s}$, $(b_{k,j})_{s\times t}$ and $(r_{k,m})_{s\times s}$ using the bases for $\mathbf{V}_h$ and W_h, from the bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and $r(\cdot,\cdot)$, respectively, and B^{T} is the transpose of matrix B. Then the matrix G is given by $(g_{k,j})_{t\times t}$, using the bases for W_h, from the term $G(p_h, q)$. This procedure is implemented on the coarse mesh for two-level methods. Our algorithms are implemented using FreeFEM^{++} [11].

In our first numerical test, the first eigenvalue of the Stokes eigenvalue problem is computed. Following [6], we employ the approximation $\lambda_1 = 52.3446911$ as the reference solution for the first eigenvalue.

We first compare our $P_2 - P_2$-stabilized method with the $P_1 - P_1$-stabilized method in [15]. From Figure 2(a), we can find our $P_2 - P_2$-stabilized method predicts $O(h^4)$ convergence rate for eigenvalue which confirms theoretical results. The $P_1 - P_1$-stabilized method has $O(h^2)$ convergence rate which is two orders lower than our method. In Table 1, we use different mesh size for the $P_2 - P_2$-stabilized method and the $P_1 - P_1$-stabilized method such that they have a similar relative error. It can be found that the CPU-time is much less for $P_2 - P_2$-stabilized method.

Then we choose the two-level discretization which satisfies $h \propto H^{3/2}$, see an example in Figure 1. In Table 2, we compare the two-level scheme with the one-level scheme (i.e. compute the eigenvalue problem on the fine mesh). The convergence rate $O(h^4)$ for both two-level and one-level schemes can be obtained. For the same h, the two-level scheme and the one-level

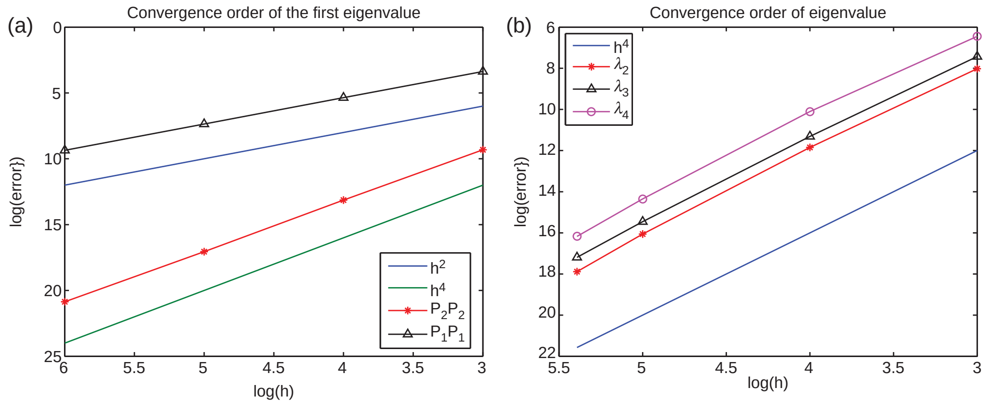

Figure 2. (a) Convergence rate by using different methods with the one-level method. (b) The convergence rate of the eigenvalue for $\lambda_{2,3,4}$ on the unit square with the two-level method.

Table 1. Comparison between two stabilized methods for the one-level method.

$P_1 - P_1$				$P_2 - P_2$			
$\frac{1}{h}$	λ_h	$\frac{\lvert\lambda-\lambda_h\rvert}{\lvert\lambda\rvert}$	CPU	$\frac{1}{h}$	λ_h	$\frac{\lvert\lambda-\lambda_h\rvert}{\lvert\lambda\rvert}$	CPU
20	53.1614	1.5602×10^{-2}	0.202	5	52.808	8.8504×10^{-3}	0.047
40	52.5489	3.9008×10^{-3}	0.764	10	52.3801	6.7666×10^{-4}	0.156
60	52.4354	1.7332×10^{-3}	2.122	15	52.3522	1.4274×10^{-4}	0.39
80	52.3957	9.7481×10^{-4}	4.665	20	52.3471	4.6542×10^{-5}	0.769
100	52.3773	6.2383×10^{-4}	8.736	25	52.3457	1.9392×10^{-5}	1.311

Table 2. Relative error and convergence rate of the one-level and two-level methods for $P_2 - P_2$ pair.

$1/H$	$1/h$	λ_h	$\frac{\lvert\lambda-\lambda_h\rvert}{\lvert\lambda\rvert}$	Rate	CPU time
10	30	52.3452	9.5546×10^{-6}		1.997
	30	52.3452	9.4666×10^{-6}		2.028
12	40	52.3449	3.0409×10^{-6}	3.9796	3.807
	40	52.3449	3.0655×10^{-6}	3.9194	4.181
14	50	52.3448	1.2478×10^{-6}	3.9919	6.131
	50	52.3448	1.2981×10^{-6}	3.8507	8.019
15	60	52.3447	6.0510×10^{-7}	3.9698	9.235
	60	52.3447	6.6006×10^{-7}	3.7098	14.227

algorithm admit the same relative error, but the CPU time for the two-level scheme is less than that taken for computing the eigenvalue problem on the fine mesh directly. And when the h is smaller, the two-level scheme reduces more computational cost.

Next numerical test is about the second, third and fourth eigenvalues $\lambda_{2,3,4}$. The reference values are computed over a fine mesh ($h = \frac{1}{64}$) and the results are: $\lambda_{2,3,4} = 92.1245411, 92.1245843, 128.209971$. Then in Figure 2(b), we exhibit the $O(h^4)$ convergence rate as has been predicted in Theorem 4.1 with the two-level method.

In Figure 3, we plot the pressure and velocity for the two-level ($1/h = 36$ and $1/H = 11$) and the one-level ($1/h = 36$) schemes. Table 3 also lists the CPU time for the two schemes. Though the stability of two schemes is obtained from Figure 3, our two-level scheme may take less time.

Table 3. CPU time of numerical solution for Stokes eigenvalue problem at $h = \frac{1}{36}$.

Numerical solution	p_h	u_{1h}	u_{2h}
Two-level method	3.167	3.152	3.089
One-level method	3.51	3.432	3.479

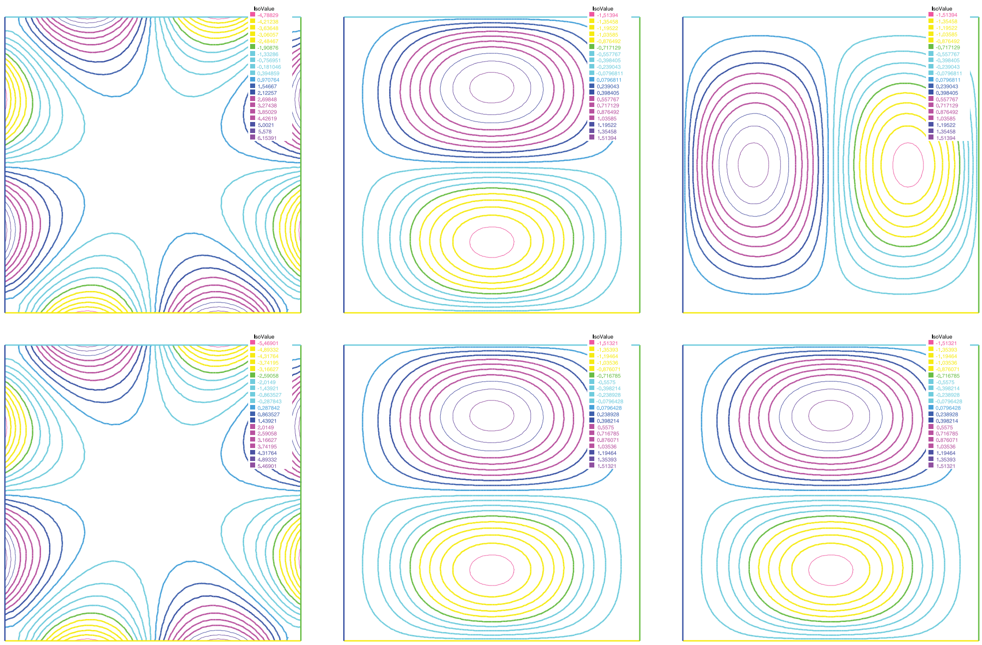

Figure 3. Plot of the pressure and velocity at $h = \frac{1}{36}$: numerical solution of the two-level method (top) and numerical solution of the one-level method (bottom) with p_h, u_{1h} and u_{2h}.

6. Conclusions

In this paper, we provided a two-level $P_2 - P_2$-stabilized finite element method based on local Gaussian quadratures to solve the Stokes eigenvalue problem. The main feature of our method is to combine the quadratic equal order stabilized method with the two-level discretization. It is shown that the given method is stable. The error estimates have been obtained, and the numerical test confirms our numerical analysis. This method can be easily extended to the Stokes eigenvalue problem in three dimensions, and other eigenvalue problems (e.g. Navier–Stokes eigenvalue problem).

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