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Numerical identification of a sparse Robin coefficient

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Abstract We investigate an inverse problem of identifying a Robin coefficient with a sparse structure in the Laplace equation from noisy boundary measurements. The sparse structure of the Robin coefficient γ is understood as a small perturbation of a reference profile γ_0 in the sense that their difference $\gamma - \gamma_0$ has a small support. This problem is formulated as an optimal control problem with an L^1-regularization term. An iteratively reweighted least-squares algorithm with an inner semismooth Newton iteration is employed to solve the resulting optimization problem, and the convergence of the iteratively weighted least-squares algorithm is established. Numerical results for two-dimensional problems are presented to illustrate the efficiency of the proposed method.

Keywords Inverse Robin problem · Sparsity regularization · Iteratively reweighted least squares method · Semismooth Newton method

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1 Introduction

In this paper, we consider the Laplace equation with mixed boundary conditions:

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = q & \text{on } \Gamma_c, \\ \frac{\partial u}{\partial n} + \gamma u = \gamma u_a & \text{on } \Gamma_i, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ is an open bounded domain with a C^1 boundary Γ. The functions q and u_a are known, which in thermal engineering refer to the heat flux and the ambient temperature, respectively. The boundary Γ consists of two disjointed parts, i.e., $\Gamma = \Gamma_c \cup \Gamma_i$, which refer to the experimentally accessible and inaccessible parts, respectively. This model can describe many physical processes, e.g., steady-state heat transfer with convective conduction occurring at the interface Γ_i between the conducting body and the ambient environment, where the third line in (1.1) is also known as Newton's law for cooling.

In several important practical scenarios, e.g., corrosion detection [18], the Robin coefficient γ represents an unknown damage profile. A similar problem arises in the nondestructive detection of metal-oxide-silicon field effect transistor (MOSFET) [11], where the Robin coefficient γ measures the quality of the metal-silicon contact. All these applications lead to the inverse Robin problem of estimating the Robin coefficient γ on the boundary Γ_i from either the data on the boundary Γ_c or internal measurements. Typically, the temperature/voltage data on the boundary Γ_c can be measured in practice, which gives the Cauchy data on the boundary Γ_c:

$$u = g \quad \text{and} \quad \frac{\partial u}{\partial n} = q \qquad \text{on } \Gamma_c.$$

The inverse Robin problem is representative in parameter identification problems for differential equations. The uniqueness of a solution to the inverse problem is ensured by unique continuation principle [19]. However, the existence of a solution requires a certain compatibility condition on the data, which is not easily verifiable. More importantly, the solution does not depend continuously on the data in the sense that small errors in the data may lead to large deviations of the solution, as a direct consequence of the severe ill-posedness of the Cauchy problem for Laplace's equation [2]. Nonetheless, due to the broad range of practical applications, there are a lot of existing works on numerical methods for the inverse Robin problem [6, 12, 13, 18, 24, 27]. Numerically, in [13, 18], the asymptotic expansion technique was employed, by assuming that the domain is a thin plate. The output least squares approach has been frequently used [12, 24, 27]. In [12, 27], the least-squares approach was implemented with an integral equation, whereas in [24], a regularized least squares approach was implemented in finite element method, and the convergence of the finite element approximation was studied. These methods are suitable for recovering smooth Robin coefficients. Recently, the case of piecewise constant Robin coefficient was considered in [23] and [6], respectively, using the Modica-Mortola functional and Kohn-Vogelius functional, which allow reconstructing sharp edges. In all these studies, the a priori information plays an essential role in obtaining useful reconstructions.

In the present work, we consider the following practical scenario. In corrosion detection, the defects occurs only at small spots of the device, and thus the difference between the profile with damages (Robin coefficient γ) and the reference profile (reference coefficient γ_0) exhibits a natural sparse structure with respect to the pixel basis. Analogously, in MOSFET [11], the imperfect contact usually occurs only locally, and thus the deviation of the physical contact profile from the reference one exhibits a natural sparse structure. Therefore, it is of much practical interest to consider the sparse case. Surely, the sparse structure should be exploited in the reconstruction procedure. However, the existing methods, including the classical L^2 regularization, cannot adequately describe the sparse structure, and hence they fail to yield satisfactory reconstructions in this case.

It is now widely accepted that the sparse structure can be effectively enforced by the so-called sparsity regularization, and the sparsity concept has received considerable attention in inverse problems community [22]. Roughly speaking, there are two different approaches to sparsity: The first takes a suitable basis and promotes sparsity by the ℓ^1-norm of the expansion coefficient, whereas the second considers the L^1-regularized problem. We shall adopt the L^1-regularization approach, as was discussed in [5, 7, 28], for the inverse Robin problem, which leads to a nonsmooth and nonconvex optimization problem. Hence its efficient numerical treatment is highly nontrivial. In case of linear inverse problems with ℓ^1-regularization, there are many efficient algorithms, e.g., iterated soft shrinkage [9], semismooth Newton method [16], iteratively reweighted least squares (IRLS) method [10]. In addition, iterated soft shrinkage has been extended to nonlinear inverse problems [4][22, Section 4].

We will apply an IRLS method to the sparsity constraint for the inverse Robin problem. The IRLS method has been very popular in statistics, especially robust estimators [3, 15, 29], and recently received revived interest in sparse recovery with linear inverse problems [10, 26]. It invokes the solution of a sequence of L^2-regularized subproblems, which, in our context, is a standard optimal control problem with the pointwise bilateral control constraints on the Robin coefficient. It is well known that the semismooth Newton method is especially efficient for such problems [20]. We apply a primal-dual active set algorithm, which is known to be equivalent to a semismooth Newton method [17] to solve the L^2 subproblems. The overall numerical algorithm is very efficient and robust since the IRLS method provides good initial guesses while the semismooth Newton method converges locally superlinearly. Furthermore, we establish the convergence of the IRLS method specifically for the inverse Robin problem. We are not aware of any convergence result of the IRLS method for nonlinear inverse problems in the literature. In addition, we also showcase the effectiveness of the approach for sparse reconstruction for the inverse Robin problem, and the efficiency of the numerical algorithm.

The rest of this paper is organized as follows. In Section 2, we describe the properties of forward operator, e.g., a prior estimate and differentiability. Then we discuss the well-posedness of the L^1 regularized model and its smoothed approximation in Section 3. The numerical algorithm and its convergence analysis are given in Section 4. Finally, numerical results for two-dimensional examples are presented in

Section 5 to illustrate the accuracy and efficiency of the proposed approach. Throughout we denote by C a generic constant which may differ at different occurrences, but it is independent of γ and u.

2 Analysis of the forward operator

Throughout, we shall assume that the sought-for Robin coefficient γ lies in the following admissible set:

$$\mathcal{A} = \left\{ \gamma \in L^2(\Gamma_i) : c_0 \le \gamma(x) \le c_1 \quad \text{a.e. } x \in \Gamma_i \right\}, \tag{2.1}$$

where c_0 and c_1 are two given positive constants, and on the set $\mathcal{A}$ we endowed it with the $L^2(\Gamma_i)$ norm. The constants c_0 and c_1 stand for the physical constraints, and are available in the literature for certain applications [25, Table 4.3.10]. Then for any Robin coefficient $\gamma \in \mathcal{A}$, the governing equation reads

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = q & \text{on } \Gamma_c, \\ \frac{\partial u}{\partial n} + \gamma u = \gamma u_a & \text{on } \Gamma_i. \end{cases} \tag{2.2}$$

Further, we assume the flux $q \in (H^{1/2}(\Gamma_c))'$ and ambient temperature $u_a \in L^2(\Gamma_i)$. Then we can define a forward operator $u : \mathcal{A} \mapsto H^1(\Omega)$, with $u = u(\gamma)$ being the solution of (2.2). Clearly, for any $\gamma \in \mathcal{A}$, there exists a unique solution $u(\gamma) \in H^1(\Omega)$ to (2.2). Hence the forward operator $u(\gamma)$ is well-posed from $\mathcal{A}$ to $H^1(\Omega)$. The weak formulation of (2.2) is given by: find $u \in H^1(\Omega)$ such that for any $v \in H^1(\Omega)$

$$\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\Gamma_i} \gamma u v \, ds = \int_{\Gamma_i} \gamma u_a v \, ds + \int_{\Gamma_c} q v \, ds.$$

In this part, we summarize some properties of the forward operator, i.e., a prior estimate and differentiability.

First we give an a priori estimate of the solution u to (2.2).

Lemma 2.1 *Assume $q \in (H^{1/2}(\Gamma_c))'$ and $u_a \in L^2(\Gamma_i)$. Then there exists a unique solution $u \in H^1(\Omega)$ to (2.2) and it satisfies the following a priori estimate*

$$\|u\|_{H^1(\Omega)} \le C(\|q\|_{(H^{1/2}(\Gamma_c))'} + \|u_a\|_{L^2(\Gamma_i)}).$$

Moreover, if $u_a \in L^r(\Gamma_i)$, $q \in L^r(\Gamma_c)$ for some $r > 2$, then there holds $u \in C(\overline{\Omega}) \cap H^1(\Omega)$, and

$$\|u\|_{C(\overline{\Omega})} \le C \left(\|q\|_{L^r(\Gamma_c)} + \|u_a\|_{L^r(\Gamma_i)} \right). \tag{2.3}$$

Proof By Lax-Milgram theorem, there exists a solution $u \in H^1(\Omega)$ and it satisfies the a priori estimate

$$\|u\|_{H^1(\Omega)} \le C \left(\|q\|_{(H^{1/2}(\Gamma_c))'} + \|u_a\|_{L^2(\Gamma_i)} \right).$$

Then by trace theorem and Sobolev embedding theorem [1], we have

$$u \in H^{1/2}(\Gamma_i) \hookrightarrow L^s(\Gamma_i), \quad \text{for} \begin{cases} s < \infty & \Omega \subset \mathbb{R}^2, \\ 2 < s \leq 4 & \Omega \subset \mathbb{R}^3. \end{cases}$$

and thus $\frac{\partial u}{\partial n}|_{\Gamma_i} = \gamma(u_a - u) \in L^r(\Gamma_i)$, i.e., the harmonic function u satisfies an $L^r(\Gamma)$ Neumann boundary condition $\frac{\partial u}{\partial n}|_\Gamma \in L^r(\Gamma)$. Consequently, there holds [14, 21] $u \in W^{1+1/r,r}(\Omega) \hookrightarrow C(\overline{\Omega})$ with

$$\|u\|_{C(\overline{\Omega})} \leq C\left(\|q\|_{L^r(\Gamma_c)} + \|\gamma(u - u_a)\|_{L^r(\Gamma_i)}\right) \leq C\left(\|q\|_{L^r(\Gamma_c)} + \|u_a\|_{L^r(\Gamma_i)}\right).$$

This completes the proof of the lemma. □

Remark 2.1 The a priori estimate $\|u\|_{C(\overline{\Omega})}$ in Lemma 2.1 can be extended to the following general case

$$\begin{cases} -\nabla \cdot (\alpha \nabla u) = f & \text{in } \Omega, \\ \alpha \frac{\partial u}{\partial n} = q & \text{on } \Gamma_c, \\ \alpha \frac{\partial u}{\partial n} + \gamma u = \gamma u_a & \text{on } \Gamma_i, \end{cases}$$

under the assumption that $f \in L^2(\Omega)$, $\alpha \in W^{1,\infty}(\Omega)$, $\alpha(x) \geq \alpha_0 > 0$, $u_a \in L^r(\Gamma_i)$, and $q \in L^r(\Gamma_c)$ for some $r > 2$. The proof is similar. Specifically, by Lax-Milgram theorem, we first obtain that

$$\|u\|_{H^1(\Omega)} \leq C(f, \alpha, q, u_a).$$

Then we rewrite the underlying elliptic equation into

$$\begin{cases} -\Delta u = \frac{f + \nabla\alpha \cdot \nabla u}{\alpha} & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = \frac{q}{\alpha} & \text{on } \Gamma_c, \\ \frac{\partial u}{\partial n} = \frac{\gamma(u_a - u)}{\alpha} & \text{on } \Gamma_i. \end{cases}$$

Upon noticing the relations

$$\frac{f + \nabla\alpha \cdot \nabla u}{\alpha} \in L^2(\Omega), \quad \frac{q}{\alpha} \in L^r(\Gamma_c), \quad \text{and} \quad \frac{\gamma(u_a - u)}{\alpha} \in L^r(\Gamma_i),$$

and by repeating the argument in Lemma 2.1, we deduce that $\|u\|_{C(\bar{\Omega})} \leq C(f, \alpha, q, u_a)$. With this estimate, the analytic results stated below can be extended to the case of a general elliptic operator.

Now we turn to the differentiability of the forward operator. To this end, let $u^1 = u'(\gamma)\xi$, with ξ being the direction of the derivative, which satisfies the sensitivity equation:

$$\begin{cases} -\Delta u^1 = 0 & \text{in } \Omega, \\ \frac{\partial u^1}{\partial n} = 0 & \text{on } \Gamma_c, \\ \frac{\partial u^1}{\partial n} + \gamma u^1 = \xi(u_a - u(\gamma)) & \text{on } \Gamma_i. \end{cases} \tag{2.4}$$

With the help of the sensitivity equation, we can show [23]:

Lemma 2.2 *The forward operator $u : \gamma \mapsto u(\gamma)$ is Fréchet differentiable in the following sense: for any γ, $\gamma + \xi \in \mathcal{A}$, we have*

$$\lim_{\|\xi\|_{L^2(\Gamma_i)}\to 0} \frac{\|u(\gamma+\xi) - u(\gamma) - u'(\gamma)\xi\|_{H^1(\Omega)}}{\|\xi\|_{L^2(\Gamma_i)}} = 0.$$

We shall apply an L^1-regularized output least squares method to the inverse Robin problem. Hence the Fréchet derivative of the functional $\frac{1}{2}\|u(\gamma)-g\|^2_{L^2(\Gamma_c)}$ is essential for constructing gradient-type or Newton-type algorithms. To arrive at an explicit expression of the derivative, we introduce the adjoint equation:

$$\begin{cases} -\Delta p = 0 & \text{in } \Omega, \\ \frac{\partial p}{\partial n} = u(\gamma) - g & \text{on } \Gamma_c, \\ \frac{\partial p}{\partial n} + \gamma p = 0 & \text{on } \Gamma_i. \end{cases} \tag{2.5}$$

With the help of Lemma 2.2, it is easy to show:

Lemma 2.3 *The functional $\frac{1}{2}\|u(\gamma) - g\|^2_{L^2(\Gamma_c)}$ is Fréchet differentiable, and its derivative is given by $(u_a - u(\gamma))p|_{\Gamma_c}$, where p solves the adjoint* Eq. 2.5.

3 L^1-regularization

Now we introduce an L^1-model for the inverse Robin problem. Let γ_o be the reference Robin coefficient. The true Robin coefficient γ is assumed to be identical with γ_o in the most parts, and thus the difference $\gamma - \gamma_o$ exhibits a sparse structure. Under such a priori information, the L^1-regularization emerges as a natural choice. This leads to the following cost functional:

$$J_\eta(\gamma) = \frac{1}{2}\|u(\gamma) - g\|^2_{L^2(\Gamma_c)} + \eta\|\gamma - \gamma_o\|_{L^1(\Gamma_i)}.$$

The first fitting term in J_η incorporates the data g, and the second term is the L^1-regularization to stabilize the formulation and enforces the sparsity of the Robin coefficient. The regularization parameter $\eta > 0$ controls the tradeoff between the two terms. Now the associated optimization problem reads:

Problem 3.1 $\min_{\gamma\in\mathcal{A}} J_\eta(\gamma)$.

Despite the fact that the L^1-space is not weakly sequentially compact, the existence of a minimizer to Problem 3.1 is ensured, due to the presence of the box constraints; further, the minimizer depends continuously on the observation data g. The proof is standard and thus omitted.

Theorem 3.1 *For any $\eta > 0$, there exists at least one minimizer to Problem* 3.1. *Further, let $\{g^n\} \subset L^2(\Gamma_c)$ be a sequence such that* $\lim_{n\to\infty} g^n \to g$ *in $L^2(\Gamma_c)$, and $\{\gamma^n\}$ be the sequence of minimizers to the perturbed functionals with g^n in place of g.*

Then every subsequence of $\{\gamma^n\}$ *contains a subsequence that converges in* $L^\infty(\Gamma_i)$*-weak* $*$ *to a minimizer of Problem* 3.1.

Since the L^1 norm is not differentiable, gradient type methods or Newton method cannot be applied to solve problem (3.1) directly. One natural idea is to smooth the functional. We follow this idea, and introduce the following smoothed approximation

$$J_{\eta,\epsilon}(\gamma) = \tfrac{1}{2}\|u(\gamma) - g\|^2_{L^2(\Gamma_c)} + \eta \int_{\Gamma_i} \sqrt{(\gamma - \gamma_o)^2 + \epsilon^2}\, ds.$$

Accordingly, the smoothed minimization problem reads:

Problem 3.2 $\min_{\gamma\in\mathcal{A}} J_{\eta,\epsilon}(\gamma)$.

We can prove the existence of a minimizer to Problem 3.2.

Theorem 3.2 *For any* $\eta > 0$*, there exists at least one minimizer to Problem* 3.2.

Proof First we show the weak lower semicontinuity of the penalty term $\int_{\Gamma_i} \sqrt{(\gamma - \gamma_o)^2 + \epsilon^2} ds$. By [8, Theorem 1.1], it suffices to show the convexity of the integrand $\sqrt{(\gamma - \gamma_o)^2 + \epsilon^2}$. The first and second derivatives of $\sqrt{(\gamma - \gamma_o)^2 + \epsilon^2}$ are given by

$$\frac{\gamma - \gamma_o}{\sqrt{(\gamma - \gamma_o)^2 + \epsilon^2}} \quad \text{and} \quad \frac{\epsilon^2}{[(\gamma - \gamma_o)^2 + \epsilon^2]^{\frac{3}{2}}},$$

respectively. Hence it follows directly that the term $\int_{\Gamma_i} \sqrt{(\gamma - \gamma_o)^2 + \epsilon^2} ds$ is convex.

Now the sketch the rest of the proof. Clearly $J_{\eta,\epsilon}$ is bounded from below by zero over the admissible set $\mathcal{A}$, and thus there exists a minimizing sequence $\{\gamma^k\} \subset \mathcal{A}$ such that $\lim_{k\to\infty} J_{\eta,\epsilon}(\gamma^k) = \inf_{\gamma\in\mathcal{A}} J_{\eta,\epsilon}(\gamma)$. In view of the box constraints, the sequence $\{\gamma^k\}$ is uniformly bounded in $L^\infty(\Gamma_i)$, by the Banach-Alaoglu-Bourbaki theorem, there exists a subsequence, also denoted by $\{\gamma^k\}$, and some γ^* such that $\gamma^k \to \gamma^*$ in $L^\infty(\Gamma_i)$ weak $*$. By the convexity and closedness of the set $\mathcal{A}$, $\gamma^* \in \mathcal{A}$. Further, by [24, Theorem 3.2] and the compact embedding from $H^1(\Omega)$ into $L^2(\Gamma)$ [1], there holds

$$u(\gamma^k) \to u(\gamma^*) \quad \text{weakly in } H^1(\Omega) \quad \text{and} \quad u(\gamma^k) \to u(\gamma^*) \quad \text{in } L^2(\Gamma).$$

This together with the weak lower semicontinuity of the penalty yields that γ^* is a minimizer to $J_{\eta,\epsilon}$. □

The next theorem shows the convergence of the minimizer with respect to the parameter ϵ.

Theorem 3.3 *Let* $\epsilon_n \to 0^+$ *and* γ_{ϵ_n} *be a minimizer of* J_{η,ϵ_n} *over* $\mathcal{A}$*. Then the sequence* $\{\gamma_{\epsilon_n}\}$ *contains a subsequence that converges in* $L^\infty(\Gamma_i)$*-weak* $*$ *to a minimizer of Problem* 3.1.

Proof Clearly $\{\gamma_{\epsilon_n}\}$ is bounded in $L^\infty(\Gamma_i)$ and contains a subsequence (still denote by $\{\gamma_{\epsilon_n}\}$) that converges to some $\hat{\gamma} \in \mathcal{A}$ in $L^\infty(\Gamma_i)$-weak $*$. Then by arguing similarly as in Theorem 3.2, we deduce

$$J_\eta(\hat{\gamma}) \leq \liminf_{n\to\infty} J_\eta(\gamma_{\epsilon_n}).$$

We observe that for any $\gamma \in \mathcal{A}$, by the triangle inequality, there holds

$$J_\eta(\gamma) \leq J_{\eta,\epsilon_n}(\gamma) = \tfrac{1}{2}\|u(\gamma)-g\|^2_{L^2(\Gamma_c)} + \eta\int_{\Gamma_i}\sqrt{|\gamma-\gamma_o|^2+\epsilon_n^2} \leq J_\eta(\gamma)+\eta\epsilon_n|\Gamma_i|.$$

Therefore, for any minimizer γ^* to Problem 3.1, there holds

$$J_\eta(\gamma^*) + \eta\epsilon_n|\Gamma_i| \geq J_{\eta,\epsilon_n}(\gamma^*) \geq J_{\eta,\epsilon_n}(\gamma_{\epsilon_n}) \geq J_\eta(\gamma_{\epsilon_n}).$$

Taking the limit as $n \to \infty$ yields $J_\eta(\hat{\gamma}) \leq J_\eta(\gamma^*)$, which completes the proof. □

4 Numerical method

Now we discuss the numerical treatment of Problem 3.2. We adopt the iteratively reweighted least-squares (IRLS) method, with the inner Newton updates. By Lemma 2.3, the functional $J_{\eta,\epsilon}$ is Fréchet differentiable with its derivative given by

$$J'_{\eta,\epsilon}(\gamma)\zeta = \int_{\Gamma_i}\zeta\left[(u_a-u(\gamma))p+\eta\frac{\gamma-\gamma_o}{\sqrt{(\gamma-\gamma_o)^2+\epsilon^2}}\right]ds, \tag{4.1}$$

where p is the adjoint variable defined in Eq. 2.5. Let us first examine a straightforward application of the Newton method. In the Hessian of $J_{\eta,\epsilon}$, the second-order derivative of the penalty functional is given by $\epsilon^2((\gamma-\gamma_o)^2+\epsilon^2)^{-3/2}$. Hence, when the quantity $(\gamma-\gamma_o)^2$ is away from zero, the second-order derivative is of order $O(\epsilon^2)$. In particular, if the initial guess (or intermediate updates) is not sparse, the penalty term is only of $O(\epsilon^2)$, and hence one needs to solve an ill-conditioned system at each Newton step. To circumvent this issue, we propose to use the IRLS method, i.e.,

$$J_{\eta,\epsilon}(\gamma,\omega) = \tfrac{1}{2}\|u(\gamma)-g\|^2_{L^2(\Gamma_c)} + \tfrac{\eta}{2}\int_{\Gamma_i}[(\gamma-\gamma_o)^2\omega+\epsilon^2\omega+\omega^{-1}]ds, \tag{4.2}$$

then to find the minimizer of the functional $J_{\eta,\epsilon}(\gamma,\omega)$ along the ω and γ directions alternatively. The algorithm is given in Algorithm 1.

Algorithm 1 The IRLS algorithm

1: Given initial guess γ^0, let $\omega^0 := [(\gamma^0-\gamma_o)^2+\epsilon^2]^{-\frac{1}{2}}$.
2: **for** $n = 0, 1, 2, \ldots$ **do**
3: Find $\gamma^{n+1} := \arg\min_{\gamma\in\mathcal{A}} J_{\eta,\epsilon}(\gamma,\omega^n)$;
4: Let $\omega^{n+1} = [(\gamma^{n+1}-\gamma_o)^2+\epsilon^2]^{-\frac{1}{2}}$;
5: **end for**

The IRLS method has been applied to sparse reconstruction earlier [10, 26], where the main goal is to avoid division by zero. In our approach, the IRLS method is to arrive at a stable system in Newton updates. Clearly, the IRLS method enjoys the monotonic decreasing property for the cost functional:

$$J(\gamma^{n+1}, \omega^{n+1}) \leq J(\gamma^{n+1}, \omega^n) \leq J(\gamma^n, \omega^n), \quad \forall n \geq 0.$$

The minimization of $J_{\eta,\epsilon}(\gamma, \omega)$ with respect to ω can be solved in closed form. Now we consider the minimization with respect to γ for a fixed ω^n.

Problem 4.1 $\min_{\gamma\in\mathcal{A}} J_{\eta,\epsilon}(\gamma, \omega^n)$.

Using Lemma 2.3 and standard arguments in optimal control, the first-order necessary Karush-Kuhn-Tucker system for Problem 4.1 reads

$$\textit{primal equation} \quad \begin{cases} -\Delta u = 0 & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial n} = q & \text{on} \quad \Gamma_c, \\ \frac{\partial u}{\partial n} + \gamma u = \gamma u_a & \text{on} \quad \Gamma_i, \end{cases} \tag{4.3}$$

$$\textit{adjoint equation} \quad \begin{cases} -\Delta p = 0 & \text{in} \quad \Omega, \\ \frac{\partial p}{\partial n} = u - g & \text{on} \quad \Gamma_c, \\ \frac{\partial p}{\partial n} + \gamma p = 0 & \text{on} \quad \Gamma_i, \end{cases} \tag{4.4}$$

$$\textit{optimal condition} \quad \gamma = P_{[c_0,c_1]}\left(\gamma_o + \frac{1}{\eta}\frac{(u-u_a)p}{\omega^n}\right), \tag{4.5}$$

where $P_{[c_0,c_1]}$ denotes the pointwise projection operator onto the interval $[c_0, c_1]$. Let γ^* be the optimal solution to Problem 3.2. Then accordingly the state variable u^* and adjoint variable p^* satisfy the following first-order optimality system:

$$\textit{primal equation} \quad \begin{cases} -\Delta u^* = 0 & \text{in} \quad \Omega, \\ \frac{\partial u^*}{\partial n} = q & \text{on} \quad \Gamma_c, \\ \frac{\partial u^*}{\partial n} + \gamma^* u^* = \gamma^* u_a & \text{on} \quad \Gamma_i, \end{cases} \tag{4.6}$$

$$\textit{adjoint equation} \quad \begin{cases} -\Delta p^* = 0 & \text{in} \quad \Omega, \\ \frac{\partial p^*}{\partial n} = u^* - g & \text{on} \quad \Gamma_c, \\ \frac{\partial p^*}{\partial n} + \gamma^* p^* = 0 & \text{on} \quad \Gamma_i, \end{cases} \tag{4.7}$$

$$\textit{optimal condition} \quad \gamma^* = P_{[c_0,c_1]}\left(\gamma_o + \tfrac{1}{\eta}(u^* - u_a)p^*(\sqrt{(\gamma^* - \gamma_o)^2 + \epsilon^2})^{-1/2}\right). \tag{4.8}$$

Next we study the convergence of the IRLS method. The convergence for linear inverse problems has been established [10, 26], but that for nonlinear inverse problems remains unclear. First we observe that for a fixed small positive constant ϵ, let

$\gamma^{n+1}, \omega^{n+1}$ be defined as in Algorithm 1. Then the optimal pair $(u^{n+1}, p^{n+1}, \gamma^{n+1})$ satisfies system (4.3)–(4.5). Recall that by Lemma 2.1, we have for any $r > 2$

$$\|u^{n+1}\|_{H^1(\Omega)\cap C(\overline{\Omega})} \leq C\left(\|q\|_{L^r(\Gamma_c)} + \|u_a\|_{L^r(\Gamma_i)}\right) \triangleq K,$$

$$\|p^{n+1}\|_{H^1(\Omega)\cap C(\overline{\Omega})} \leq C\|u^{n+1} - g\|_{L^r(\Gamma_c)} \triangleq K^n.$$

To obtain a local convergence result, we make the following assumption:

Assumption 4.1 (*a*) *The iterate γ^{n+1} satisfies $c_0 < \gamma^{n+1}(x) < c_1$ for all n.*
(*b*) *There exists a constant $s \in (0,1)$ such that $\frac{1}{\eta}KK^n \leq s$ for all n, and γ_o and u_a are constants.*

Under Assumption 4.1, we are ready to show our main theoretical result, i.e., the convergence of the IRLS algorithm for the inverse Robin problem.

Theorem 4.1 *Assume that $q \in L^r(\Gamma_c)$ and $u_a \in L^r(\Gamma_i)$ for some $r > 2$. Further, let Assumption 4.1 be fulfilled, and $\gamma^0 = \gamma_o$, and the sequence $\{(u^n, p^n, \gamma^n)\}$ be generated by the IRLS method. Then there exists a subsequence of $\{(u^n, p^n, \gamma^n)\}$ that converges to (u^*, p^*, γ^*) weakly in $H^1(\Omega) \times H^1(\Omega) \times L^2(\Gamma_i)$, and the limit (u^*, p^*, γ^*) satisfies (4.6)–(4.8).*

Proof By Assumption 4.1(a) and noting the defining relation $\omega^n = [(\gamma^n - \gamma_o)^2 + \epsilon^2]^{-\frac{1}{2}}$, the optimality condition (4.5) can be rewritten as

$$\gamma^{n+1} - \gamma_o = \frac{1}{\eta}(u^{n+1} - u_a)p^{n+1}\sqrt{(\gamma^n - \gamma_o)^2 + \epsilon^2}.$$

Let $s_{n+1} := \frac{1}{\eta}(u^{n+1} - u_a)p^{n+1}$. Then there holds

$$(\gamma^{n+1} - \gamma_o)^2 = s_{n+1}^2(\gamma^n - \gamma_o)^2 + \epsilon^2 s_{n+1}^2.$$

Now Assumption 4.1(b) implies

$$\gamma^{n+1} - \gamma_o = \sqrt{\epsilon^2(s_{n+1}^2 + s_{n+1}^2 s_n^2 + \cdots + s_{n+1}^2 s_n^2 \cdots s_1^2)},$$

and

$$\widehat{\omega}^n = (\omega^n)^{-1} = \sqrt{(\gamma^n - \gamma_o)^2 + \epsilon^2} = \sqrt{\epsilon^2(1 + s_n^2 + s_n^2 s_{n-1}^2 + \cdots + s_n^2 s_{n-1}^2 \ldots s_1^2)}. \tag{4.9}$$

Since u^{n+1} and p^{n+1} are defined over the whole domain Ω, we may extend the definition of γ^{n+1} and $\hat{\omega}^n$ to the whole domain Ω (with the extensions still denoted by γ^{n+1} and $\widehat{\omega}^n$). Since γ_o is a constant, their derivatives are given by

$$\nabla\gamma^{n+1} = \epsilon \frac{s_{n+1}^2 \frac{\nabla s_{n+1}}{s_{n+1}} + \cdots + s_{n+1}^2 s_n^2 \cdots s_1^2 (\frac{\nabla s_{n+1}}{s_{n+1}} + \cdots + \frac{\nabla s_1}{s_1})}{\sqrt{s_{n+1}^2 + s_{n+1}^2 s_n^2 + \cdots + s_{n+1}^2 s_n^2 \cdots s_1^2}}$$

and

$$\nabla\widehat{\omega}^n = \epsilon\frac{s_{n+1}^2\frac{\nabla s_{n+1}}{s_{n+1}} + \cdots + s_{n+1}^2 s_n^2\cdots s_1^2(\frac{\nabla s_{n+1}}{s_{n+1}} + \cdots + \frac{\nabla s_1}{s_1})}{\sqrt{1 + s_{n+1}^2 + s_{n+1}^2 s_n^2 + \cdots + s_{n+1}^2 s_n^2\cdots s_1^2}}.$$

Next we consider γ^{n+1} only, and $\widehat{\omega}^n$ can be treated similarly. Note that

$$\|\nabla\gamma^{n+1}\|_{L^2(\Omega)}^2 = \epsilon^2\int_\Omega \frac{\left[s_{n+1}^2\frac{\nabla s_{n+1}}{s_{n+1}} + \cdots + s_{n+1}^2 s_n^2 ... s_1^2(\frac{\nabla s_{n+1}}{s_{n+1}} + ... + \frac{\nabla s_1}{s_1})\right]^2}{s_{n+1}^2 + s_{n+1}^2 s_n^2 + ... + s_{n+1}^2 s_n^2 ... s_1^2}dx$$

$$= \epsilon^2\int_\Omega \frac{\left[a_{n+1}\nabla s_{n+1} + \cdots + a_1\nabla s_1\right]^2}{s_{n+1}^2 + s_{n+1}^2 s_n^2 + ... + s_{n+1}^2 s_n^2 ... s_1^2}dx,$$

where

$$\begin{cases} a_{n+1} = s_{n+1}(1 + s_n^2 + \cdots + s_n^2 s_{n-1}^2\cdots s_1^2),\\ \quad a_n = s_{n+1}^2 s_n(1 + s_{n-1}^2 + \cdots + s_{n-1}^2\cdots s_1^2),\\ \quad \cdots\\ \quad a_1 = s_{n+1}^2 s_n^2\cdots s_1. \end{cases}$$

By the Cauchy-Schwarz inequality, we obtain

$$\|\nabla\gamma^{n+1}\|_{L^2(\Omega)}^2 \le \epsilon^2\int_\Omega \frac{\left(a_{n+1}^2 + 2^2 a_n^2 + \cdots + (n+1)^2 a_1^2\right)\left(|\nabla s_{n+1}|^2 + |\frac{\nabla s_n}{2}|^2 + \cdots + |\frac{\nabla s_1}{n+1}|^2\right)}{s_{n+1}^2 + s_{n+1}^2 s_n^2 + ... + s_{n+1}^2 s_n^2 ... s_1^2}dx$$
$$= \epsilon^2\int_\Omega \left(b_{n+1} + 2^2 b_n + \cdots + (n+1)^2 b_1\right)\left(|\nabla s_{n+1}|^2 + |\frac{\nabla s_n}{2}|^2 + \cdots + |\frac{\nabla s_1}{n+1}|^2\right)dx,$$

where

$$\begin{cases} b_{n+1} = \dfrac{a_{n+1}^2}{s_{n+1}^2 + s_{n+1}^2 s_n^2 + ... + s_{n+1}^2 s_n^2 ... s_1^2} = 1 + s_n^2 + \cdots + s_n^2 s_{n-1}^2\cdots s_1^2,\\ \quad b_n = \dfrac{a_n^2}{s_{n+1}^2 + s_{n+1}^2 s_n^2 + ... + s_{n+1}^2 s_n^2 ... s_1^2} \le s_{n+1}^2 s_n^2(1 + s_{n-1}^2 + \cdots + s_n^2 s_{n-1}^2\cdots s_1^2),\\ \quad \cdots\\ \quad b_1 = \dfrac{a_1^2}{s_{n+1}^2 + s_{n+1}^2 s_n^2 + ... + s_{n+1}^2 s_n^2 ... s_1^2} \le s_{n+1}^2 s_n^2\cdots s_2^2 s_1^2. \end{cases}$$

By Assumption 4.1(b), there holds $\|s_i\|_{L^\infty(\Omega)} \le s$ and consequently, $\|b_i\|_{L^\infty(\Omega)} \le \frac{s^{2n+2-2i}}{1-s^2}$ for any $1 \le i \le n+1$. Similarly

$$\|\nabla s_i\|_{L^2(\Omega)} = \frac{1}{\eta}\|(\nabla u^{n+1})p^{n+1} + (\nabla p^{n+1})u^{n+1}\|_{L^2(\Omega)}$$
$$\le \frac{1}{\eta}\left(\|u^{n+1}\|_{H^1(\Omega)}\|p^{n+1}\|_{L^\infty(\Omega)} + \|p^{n+1}\|_{H^1(\Omega)}\|u^{n+1}\|_{L^\infty(\Omega)}\right) \le 2s.$$

Therefore we have

$$
\begin{aligned}
\|\nabla\gamma^{n+1}\|^2_{L^2(\Omega)} &\le \epsilon^2 \left(\|b_{n+1}\|_{L^\infty(\Omega)} + \cdots + (n+1)^2\|b_1\|_{L^\infty(\Omega)}\right) \\
&\quad \times \left(\|\nabla s_{n+1}\|^2_{L^2(\Omega)} + \cdots + \frac{1}{(n+1)^2}\|\nabla s_1\|^2_{L^2(\Omega)}\right) \\
&\le \epsilon^2 \left[\frac{1}{s^2(1-s^2)^2}\sum_{i=1}^{\infty} s^{2i} i^2\right]\left[4s^2 \sum_{i=1}^{\infty}\frac{1}{i^2}\right] \le C.
\end{aligned}
$$

Similarly, one can show that the quantities $\|\gamma^n\|_{L^2(\Omega)}$, $\|\nabla\widehat{\omega}^n\|_{L^2(\Omega)}$ and $\|\widehat{\omega}^n\|_{L^2(\Omega)}$ are also uniformly bounded. Together with a priori estimates from Lemma 2.1, we deduce that the sequence $\{(u^n, p^n, \gamma^n, \widehat{\omega}^n)\}$ is uniformly bounded in $(H^1(\Omega))^4$. Therefore, there exists a subsequence, still denoted by $\{(u^n, p^n, \gamma^n, \widehat{\omega}^n)\}$, such that it converges to some $(u^*, p^*, \gamma^*, \widehat{\omega}^*) \in (H^1(\Omega))^4$ weakly in $(H^1(\Omega))^4$. Note that the tuple $(u^n, p^n, \gamma^n, \widehat{\omega}^n)$ satisfies

$$
\textit{primal equation} \qquad \begin{cases} -\Delta u^n = 0 & \text{in} \quad \Omega, \\ \frac{\partial u^n}{\partial n} = q & \text{on} \quad \Gamma_c, \\ \frac{\partial u^n}{\partial n} + \gamma^n u^n = \gamma^n u_a & \text{on} \quad \Gamma_i, \end{cases}
$$

$$
\textit{adjoint equation} \qquad \begin{cases} -\Delta p^n = 0 & \text{in} \quad \Omega, \\ \frac{\partial p^n}{\partial n} = u^n - g & \text{on} \quad \Gamma_c, \\ \frac{\partial p^n}{\partial n} + \gamma^n p^n = 0 & \text{on} \quad \Gamma_i, \end{cases}
$$

$$
\textit{optimal condition} \qquad \eta(\gamma^n - \gamma_o) = (u^n - u_a)p^n\widehat{\omega}^n.
$$

We need to verify the limit (u^*, p^*, γ^*) satisfy (4.6)–(4.8). Upon passage to the limit, all the linear part naturally hold true. So we focus on the nonlinear part, i.e., the Robin boundary condition on Γ_i and optimality condition. By Sobolev embedding theorem [1]

$$
H^1(\Omega) \hookrightarrow H^{1/2}(\Gamma_i) \hookrightarrow L^2(\Gamma_i),
$$

where the latter embedding is compact. Then

$$
\gamma^n \to \gamma^*, \quad u^n \to u^*, \quad p^n \to p^* \ \text{ in } L^2(\Gamma_i),
$$

hence the boundary condition is also satisfied. For the optimality condition, we notice that

$$
H^{1/2}(\Gamma_i) \hookrightarrow H^{1/2-t}(\Gamma_i)
$$

is compact for any $1/2 \ge t > 0$. We may choose a sufficiently small t to ensure $H^{1/2-t}(\Gamma_i) \hookrightarrow L^3(\Gamma_i)$ when $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ and hence $H^{1/2}(\Gamma_i) \hookrightarrow L^3(\Gamma_i)$ compactly. Then the limit functions also satisfies the optimality condition:

$$
\eta(\gamma^* - \gamma_o) = (u^* - u_a)p^*\widehat{\omega}^*.
$$

It remains to show that $(\widehat{\omega}^*)^2 = (\gamma^* - \gamma_o)^2 + \epsilon^2$. This can be done by revisiting the definition in Eq. 4.9 of $\widehat{\omega}^n$. Lastly by Assumption 4.1(a), the limit function γ^* fulfills the box constraints. This completes the proof. □

Remark 4.1 We would like to comment the assumption of this theorem.

(a) Assumption 4.1(a) can be numerically verified. According to our experiences, when the constants c_0 and c_1 are chosen far from the true solution, then the constraints are never active throughout the iteration.
(b) The conditions γ_o, u_a be constants and $\gamma^0 = \gamma_o$ are not essential. The theorem can be proved without these conditions. We make these assumption to avoid the tedious computation and present the main idea of the proof.
(c) The condition $\frac{1}{\eta} K K^n \leq s$ in Assumption 4.1(b) is slightly more restrictive. It holds when η is sufficiently large or the iterate γ^n is closed to the true solution and the noise level is very small.

Remark 4.2 It is worth noting that the proof of Theorem 4.1 does not rely on the sparsity of the solution, even though sparsity is build into the regularization model. This is not surprising, in view of the fact that the IRLS algorithm is general-purposed, and it is applicable to a wide variety of optimization problems, including sparsity optimization.

To solve the L^2-subproblems arising from the IRLS method, we propose a semismooth Newton method. More specifically, given a bounded positive weight ω, we consider the following optimization problem:

$$\min_{\gamma \in \mathcal{A}} \frac{1}{2}\|u(\gamma) - g\|^2_{L^2(\Gamma_c)} + \frac{\eta}{2}\int_{\Gamma_i} \omega|\gamma|^2. \tag{4.10}$$

By introducing the Lagrange multipliers λ corresponding to the pointwise constraints for the Robin coefficient and rewriting the complementary condition into a nonlinear equation using the max and min functions, following [20], its first-order necessary condition is given by

$$\textit{primal equation} \qquad \begin{cases} -\Delta u = 0 & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial n} = q & \text{on} \quad \Gamma_c, \\ \frac{\partial u}{\partial n} + \gamma u = \gamma u_a & \text{on} \quad \Gamma_i, \end{cases} \tag{4.11}$$

$$\textit{adjoint equation} \qquad \begin{cases} -\Delta p = 0 & \text{in} \quad \Omega, \\ \frac{\partial p}{\partial n} = u - g & \text{on} \quad \Gamma_c, \\ \frac{\partial p}{\partial n} + \gamma p = 0 & \text{on} \quad \Gamma_i, \end{cases} \tag{4.12}$$

$$\textit{optimal condition} \qquad \eta\omega(\gamma - \gamma_o) - (u - u_a)p + \lambda = 0, \tag{4.13}$$

$$\textit{complementary condition} \qquad \max(0, \lambda + d(\gamma - c_1)) + \min(0, \lambda + d(\gamma - c_0)) - \lambda = 0, \tag{4.14}$$

where $d > 0$ is a fixed positive constant.

We adopt a semismooth Newton method to solve the first-order necessary optimality system. To take care of the max and min functions in the complementary condition, we here use the well-established primal-dual active set technique [20]. A complete description is given in Algorithm 2. Each iteration of the semismooth Newton method requires solving one coupled linear system in the state variable u, adjoint variable p, Robin coefficient γ and Lagrangian multiplier λ.

Algorithm 2 Semismooth Newton algorithm

Set initial guess γ_0, λ^0, u^0, p^0 and the positive constant d.
for $k = 0, 1, 2, \ldots$ **do**
  compute the active sets $\mathcal{A}^{\beta}_{k+1}$ and $\mathcal{A}^{\alpha}_{k+1}$

$$\mathcal{A}^{\alpha}_{k+1} = \{x : \lambda^k + d(\gamma^k - c_0) < 0\} \quad \text{and} \quad \mathcal{A}^{\beta}_{k+1} = \{x : \lambda^k + d(\gamma^k - c_1) > 0\}.$$

  solve the following system

$$\begin{cases} \begin{cases} -\Delta u_{k+1} = 0 \quad \text{in} \quad \Omega, \\ \frac{\partial u_{k+1}}{\partial n} = q \quad \text{on} \quad \Gamma_c, \\ \frac{\partial u_{k+1}}{\partial n} + \gamma_k(u_{k+1} - u_k) + \gamma_{k+1}(u_k - u_a) = 0 \quad \text{on} \quad \Gamma_i, \end{cases} \\ \begin{cases} -\Delta p_{k+1} = 0 \quad \text{in} \quad \Omega, \\ \frac{\partial p_{k+1}}{\partial n} = u_{k+1} - g \quad \text{on} \quad \Gamma_c, \\ \frac{\partial p_{k+1}}{\partial n} + \gamma_k p_{k+1} + \gamma_{k+1} p_k - \gamma_k p_k = 0 \quad \text{on} \quad \Gamma_i, \end{cases} \\ \eta\omega\gamma_{k+1} + \lambda_{k+1} = \eta\omega_k\gamma_o + (u_{k+1} - u_k)p_k + p_{k+1}(u_k - u_a), \\ d(\chi_{\mathcal{A}^{\beta}_{k+1}} + \chi_{\mathcal{A}^{\alpha}_{k+1}})(\gamma^{k+1} - \gamma^k) + (\chi_{\mathcal{A}^{\beta}_{k+1}} + \chi_{\mathcal{A}^{\alpha}_{k+1}} - 1)(\lambda^{k+1} - \lambda^k) \\ \qquad = \max(0, \lambda^k + d(\gamma^k - c_1)) + \min(0, \lambda^k + d(\gamma^k - c_0)) - \lambda^k. \end{cases}$$

  check stop criteria.
end for

Finally we briefly comment on Algorithm 1. We remark that it is very robust and efficient since the IRLS method provides good initial guesses and the semismooth Newton method enjoys locally superlinear convergence. The algorithm involves two loops, i.e., the outer IRLS iteration and inner Newton iteration. Due to its locally superlinear convergence property, the inner Newton method converges very fast. In practice, we may fix the number of Newton iterations at one. The number of IRLS iterations is more delicate. Clearly, more IRLS iterations can yield a more accurate solution, but at the expense of increased computational cost. However, in practice, one usually adopts a continuation strategy in the regularization parameter η, in order to determine η iteratively, e.g., with the discrepancy principle. Then the minimizer of J_{η_s} only serves as an initial guess of the next subproblem $\min_{\gamma\in\mathcal{A}} J_{\eta_{s+1}}(\gamma)$, hence it is unnecessary to solve $\min_{\gamma\in\mathcal{A}} J_{\eta_s}(\gamma)$ to very high accuracy. We will illustrate this issue numerically in the next section.

5 Numerical examples and discussions

Now we present several examples to illustrate the efficiency of our approach. We consider two domains, i.e., a unit square $\Omega = (0, 1) \times (0, 1)$ and a rectangle $\widehat{\Omega} = (0, 3) \times (0.5, 1)$. The boundary Γ_i is the top side of the domain, and the remaining boundary, i.e., $\Gamma \setminus \Gamma_i$, is Γ_c. The domain Ω is triangulated into a 50×50 uniform

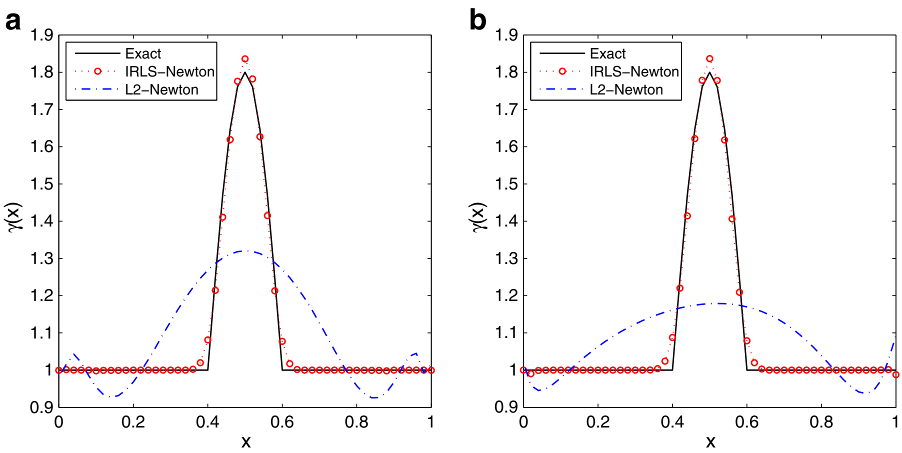


Fig. 1 The reconstructions for Example 5.1 with (a) 0% noise and (b) 0.1% noise

mesh, and $\widehat{\Omega}$ into a 150×30 uniform mesh. The conforming piecewise quadratic finite element space is adopted for discretizing both the state and adjoint problems. In practice, the data are experimentally measured and contain errors. The simulated noisy data are generated pointwise by

$$\bar{g}(x) = g^{\dagger}(x)(1 + \delta\xi) \qquad \text{on } \Gamma_c, \tag{5.1}$$

where δ denotes the noise level, and ξ is a uniform random variable in $[-1, 1]$. The function $g^{\dagger}$ refers to the exact data, and we denote by $\gamma^{\dagger}$ the true Robin coefficient. All the computations were carried out on a personal laptop. First we compare the reconstructions by the proposed L^1-model and the more conventional L^2-model [24], i.e., $J_\eta(\gamma) = \frac{1}{2}\|u(\gamma) - g\|^2_{L^2(\Gamma_c)} + \frac{\eta}{2}\|\gamma - \gamma_o\|^2_{L^2(\Gamma_i)}$. The associated optimization problem for the L^2-model was solved by a semismooth Newton algorithm, cf. Algorithm 2 with $\omega = 1$.

Example 5.1 The domain is taken to be Ω. The problem parameters are fixed at $u_a = 1.5$, $\gamma_o = 1$, and the flux $q = 2$ on the right side of the domain and vanishes elsewhere. The true Robin coefficient $\gamma^{\dagger}(x)$ is given by $\gamma^{\dagger} = \gamma_o + \frac{4}{5}\sin(5\pi (x - \frac{2}{5}))\chi_{[0.4,0.6]}$.

We carry out two tests with Example 5.1. In the first test, the data g is exact. We fix the regularization parameter at 4.0×10^{-12} for both the L^1- and L^2-models. The reconstructions are presented in Fig. 1a. Clearly, the L^1 solution agrees well with the true solution $\gamma^{\dagger}$, whereas the L^2 solution only partly captures the rough shape of the bump of the solution but it is inaccurate otherwise and hence not acceptable. In the second test, we add 0.1% noise into the data, and choose the regularization parameter in both the L^1-model and L^2-model by the discrepancy principle. The numerical

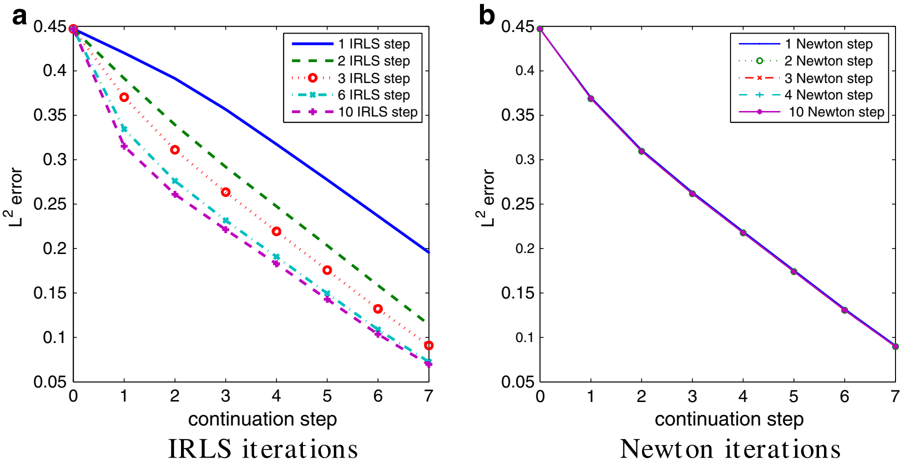


Fig. 2 Numerical results for Example 5.2 with exact data by Algorithm 1 with a continuation on the regularization parameter η: $\eta_0 = 5 \times 10^{-2}$ and $\eta_{s+1} = 0.5\eta_s$, $s = 0, 1, 2, \ldots, 6$

solutions are presented in Fig. 1b. Clearly the L^1-model yields a better reconstruction than the L^2-model.

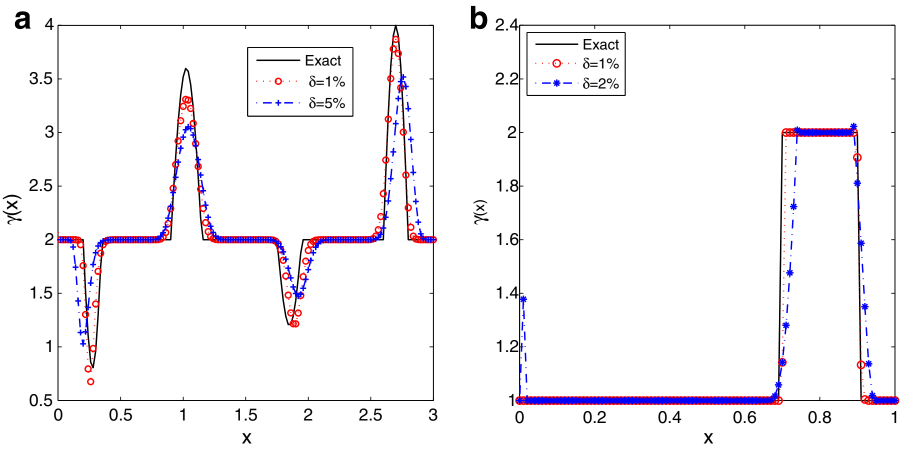
Now we illustrate the performance of Algorithm 1.

Example 5.2 The domain is taken to be $\widehat{\Omega}$. The problem parameters are fixed at $u_a = 4$, and $\gamma_o = 2$. The flux q is respectively 3, 5 and 2 on the left, bottom and right sides of the domain. The true Robin coefficient $\gamma^\dagger$ is given by $\gamma^\dagger = \gamma_o + \sin(\frac{5\pi}{2}(x - \frac{3}{10}))\chi_{[0.3,0.7]}$.

The results depend only very mildly on the number of IRLS iterations; see Fig. 2a for the L^2-error. The number of inner Newton iterations practically has no influence on the solution accuracy, cf. Fig. 2b. This is attributed to the locally superlinear convergence of the Newton method. Therefore, in later numerical tests we have taken three IRLS iterations and one Newton iteration.

Before presenting further examples, we remark that with minor changes, the algorithm can be adapted straightforwardly to the general second-order elliptic equation $-\nabla \cdot (\alpha \nabla u) = f$, cf. Remark 2.1. The only change to Algorithm 2 is to replace the Laplace operator with the general elliptic operator. We illustrate this general case in our last two examples. The next example considers a more complex profile.

Example 5.3 The domain is taken to be $\widehat{\Omega}$. The problem parameters are given by $\alpha(x) = e^x$, $f = -e^x(2x + 2e^x + 4 + e^y)$, $u_a = x^2 + e^x + y^2 + e^y + 3$, and $\gamma_o = 2$. The flux q is respectively -1, $-2e^x$ and e^6 on the left, bottom and right sides of the domain, respectively. We consider one Robin coefficient $\gamma^\dagger$ with four local bumps, i.e., $\gamma^\dagger = \gamma_o - \frac{6}{5}\sin\frac{10}{3}\pi(x - .2)\chi_{[.2,.35]} + \frac{8}{5}\sin 4\pi(x - .9)\chi_{[.9,1.15]} - \frac{4}{5}\sin 5\pi(x - 1.75)\chi_{[1.75,1.95]} + 2\sin 5\pi(x - 2.6)\chi_{[2.6,2.8]}$.



Example 5.3 Example 5.4

Fig. 3 The numerical reconstructions for two levels of noises

The results are shown in Fig. 3a. All the locations of the bumps are correctly identified, and their magnitudes are also reasonable. The algorithm is stable for up to 5 % noise in the data.

Our last example considers a nonsmooth coefficient.

Example 5.4 The domain is taken to be Ω. The problem parameters are fixed at $\alpha(x) = 1$, $f = -2$, $u_a = 1$, $\gamma_o = 1$, and the flux $q = 2$ on the right side of the domain and zero elsewhere. The true Robin coefficient $\gamma^\dagger$ is discontinuous and it is given by $\gamma^\dagger = \gamma_o + \chi_{[0.7,0.9]}$,and we set $c_0 = 1$, $c_1 = 2$ here.

As is expected, the proposed algorithm can stably and accurately identify the distinct feature of the discontinuous coefficient in the presence of noises, cf. Fig. 3b. We observe that at the left end point, there is one spurious oscillation. We would like to note that such localized spurious oscillations seem common in the reconstructions by the semismooth Newton method, especially in the case of large noise levels. However, a theoretical justification of the observation is still missing.

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