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East Asian Journal on Applied Mathematics / Volume 5 / Issue 03 / August 2015, pp 273 - 300
DOI: 10.4208/eajam.310315.030715a, Published online: 07 September 2015

East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.310315.030715a

Vol. **5**, No. 3, pp. 273-300
August 2015

Tikhonov Regularisation Method for Simultaneous Inversion of the Source Term and Initial Data in a Time-Fractional Diffusion Equation

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Received 31 March 2015; Accepted (in revised version) 3 July 2015.

Abstract. The inverse problem of identifying the time-independent source term and initial value simultaneously for a time-fractional diffusion equation is investigated. This inverse problem is reformulated into an operator equation based on the Fourier method. Under a certain smoothness assumption, conditional stability is established. A standard Tikhonov regularisation method is proposed to solve the inverse problem. Furthermore, the convergence rate is given for an *a priori* and *a posteriori* regularisation parameter choice rule, respectively. Several numerical examples, including one-dimensional and two-dimensional cases, show the efficiency of our proposed method.

AMS subject classifications: 65N21, 49M15

Key words: Time-fractional diffusion equation, conditional stability, Tikhonov regularisation, Morozov discrepancy principle, convergence rate.

1. Introduction

Let Ω denote an open bounded domain in $\mathbb{R}^d$ $(d = 1,2,3)$ with sufficiently smooth boundary $\partial\Omega$, and let us consider the following time-fractional diffusion equation with homogeneous Dirichlet boundary condition:

$$\begin{cases} \dfrac{\partial^\alpha u(x,t)}{\partial t^\alpha} + Lu(x,t) = f(x)\,, & (x,t) \in \Omega \times (0,T)\,, \\ u(x,t)|_{\partial\Omega} = 0\,, & (x,t) \in \partial\Omega \times (0,T]\,, \\ u(x,0) = u^0(x)\,, & x \in \Omega\,, \end{cases} \tag{1.1}$$

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where $\alpha \in (0,1)$ and L is a symmetric strongly elliptic operator given by

$$L(u) = -\sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{d} \theta_{i,j} \frac{\partial}{\partial x_j} u(x)\right) + c(x)u(x)\,.$$

Assume that

$$\theta_{i,j} \in C^1(\overline{\Omega}),\ \theta_{i,j} = \theta_{j,i}, c(x) \in C(\overline{\Omega}),\ c(x) \geq 0, \forall x \in \overline{\Omega}\,,$$

and there exists a constant $\nu > 0$ such that $\nu \sum_i^d \xi_i^2 \leq \sum_{i,j=1}^d \theta_{i,j}(x)\xi_i\xi_j\ \forall x \in \overline{\Omega},\ \xi \in \mathbb{R}^d$. The fractional derivative $\partial^\alpha u(x,t)/\partial t^\alpha$ is the Caputo fractional derivative defined by

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\eta)^{-\alpha} \frac{\partial u}{\partial \eta} \mathrm{d}\eta,\ 0 < \alpha < 1\,, \tag{1.2}$$

where $\Gamma(1-\alpha)$ is the Gamma function.

The time-fractional diffusion equation (1.1) has received much attention recently, due to many applications in various areas of engineering. The mathematical theory and the associated numerical method for the anomalous diffusion equation have often been discussed — e.g. see [1,5,9,10,13–15] and references therein. The inverse problem for the time-fractional diffusion equation has also been studied extensively. For example, the backward problem was explored by the quasi-reversibility method, the optimisation method, the data regularisation method, and the spectral truncation method [11,17,18,22,28]. In Refs. [26,29], the eigenfunction expansion and integral equation methods were applied to recover the space-dependent or time-dependent source term, respectively. The inverse boundary problem, inverse potential problem, inverse coefficient problem and the order identification problem have all been investigated [6,8,12,27].

Here we consider the reconstruction of the source term $f(x)$ and initial condition $u(x,0) = u^0(x)$ from the noisy measurement $g_i^\delta \approx u(x,T_i),\ i = 1,2$ with $0 < T_1 < T_2$ and the noise level such that

$$\left\|u(x,T_1) - g_1^\delta\right\| \leq \delta\,, \qquad \left\|u(x,T_2) - g_2^\delta\right\| \leq \delta\,. \tag{1.3}$$

When $\alpha = 1$, this is the simultaneous identification problem for the diffusion equation, which has been studied using various approaches [7,24,25,30].

Some work has been done on the convergence analysis for Tikhonov regularisation of the backward problem or the source inverse problem. For example, in Ref. [20] the Tikhonov regularisation method was provided to solve a backward problem for a time-fractional diffusion equation, and the respective convergence rates were obtained for an *a priori* or an *a posteriori* regularisation parameter choice rule. A space-dependent source identification problem and corresponding convergence estimates for the Tikhonov regularisation method have been considered [21]. However, to the best of our knowledge the problem of simultaneously identifying the initial data and the source term for the time-fractional diffusion equation has not been solved previously.

Some preliminary results are given in Section 2. In Section 3, the ill-posedness of the inverse problem and a conditional stability result are provided. In Section 4, the Tikhonov regularisation method is applied and convergence rates under both *a priori* and *a posterior* parameter choice rules are given. Numerical methods, and several numerical examples to validate our approach, are presented In Section 5. Our brief conclusion is in Section 6, and some Lemmas and Theorems are proven in the appendices. Throughout, C denotes a generic constant, which may differ from place to place but is always independent of the noise level δ.

2. Preliminaries

Given arbitrary constants $a>0$ and $b\in\mathbb{R}$, the Mittag-Leffler function is defined by

$$E_{a,b}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(ak+b)}\,,\qquad z\in\mathbb{C}\,,$$

and has properties as follows [14].

Lemma 2.1. **(a)** *For* $0<\alpha<1$ *and* $\eta>0$*,*

$$0\le E_{\alpha,1}(-\eta)<1\,,\qquad \frac{\mathrm{d}^\alpha}{\mathrm{d}\eta^\alpha}E_{\alpha,1}(-\lambda\eta^\alpha)=-\lambda E_{\alpha,1}(-\lambda\eta^\alpha)\,.$$

Moreover, $E_{\alpha,1}(-\eta)$ *is fully monotonic — i.e.* $(-1)^n\mathrm{d}^{\mathrm{n}}E_{\alpha,1}(-\eta)/\mathrm{d}\eta^{\mathrm{n}}\ge 0$. *When* $\eta\to+\infty$*,* $E_{\alpha,1}(-\eta)$ *satisfies the following approximation relation:*

$$E_{\alpha,1}(-\eta)=\frac{1}{\eta\Gamma(1-\alpha)}+\mathscr{O}\left(|\eta|^{-2}\right).$$

(b) *For* $\lambda>0$*,* $\alpha>0$ *and positive integer* $m\in\mathbb{N}$*,*

$$\frac{\mathrm{d}^m}{\mathrm{d}t^m}E_{\alpha,1}(-\lambda t^\alpha)=-\lambda t^{\alpha-1}E_{\alpha,\alpha-m+1}(-\lambda t^\alpha)\,,\quad t>0\,.$$

(c) *Assume that* $0<\alpha_0<\alpha_1<1$*. Then there are constants* $c,\ C>0$*, which are only relative to* α_0 *and* α_1*, such that*

$$\frac{c}{\Gamma(1-\alpha)}\frac{1}{1-t}\le E_{\alpha,1}(t)\le\frac{C}{\Gamma(1-\alpha)}\frac{1}{1-t}\quad\forall t\le 0\,,$$

uniformly for all $\alpha\in[\alpha_0,\alpha_1]$.

By superposition, the solution $u(x,t)$ satisfying the problem (1.1) can be decomposed into the components $u_1(x,t)$ and $u_2(x,t)$ that satisfy two subproblems — i.e.

$$\begin{cases}\dfrac{\partial^\alpha u_1(x,t)}{\partial t^\alpha}+Lu_1(x,t)=f(x)\,, & (x,t)\in\Omega\times(0,T)\,,\\ u_1(x,t)|_{\partial\Omega}=0\,, & (x,t)\in\partial\Omega\times(0,T]\,,\\ u_1(x,t)|_{t=0}=0\,, & x\in\Omega\,,\end{cases}\tag{2.1}$$

$$\begin{cases} \dfrac{\partial^\alpha u_2(x,t)}{\partial t^\alpha} + Lu_2(x,t) = 0\,, & (x,t) \in \Omega \times (0,T)\,, \\ u_2(x,t)|_{\partial\Omega} = 0\,, & (x,t) \in \partial\Omega \times (0,T]\,, \\ u_2(x,t)|_{t=0} = u^0(x)\,, & x \in \Omega\,, \end{cases} \tag{2.2}$$

such that

$$u(x,t) = u_1(x,t) + u_2(x,t)\,. \tag{2.3}$$

Since L is a symmetric strongly elliptic operator, if $\lambda_k \in \mathbb{R}$ and $\varphi_k(x) \in H^2(\Omega) \cap H_0^1(\Omega)$ denote its eigenvalues and corresponding orthonormal eigenfunctions we have

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots\,, \qquad \lim_{k\to\infty} \lambda_k = +\infty\,. \tag{2.4}$$

Applying separation of variables and Lemma 2.1 then yields the formal solutions of (2.1) and (2.2) — viz.

$$u_1(x,t) = \sum_{k=1}^{+\infty} f_k \frac{1 - E_{\alpha,1}(-\lambda_k t^\alpha)}{\lambda_k} \varphi_k(x)\,, \tag{2.5}$$

$$u_2(x,t) = \sum_{k=1}^{+\infty} E_{\alpha,1}(-\lambda_k t^\alpha) u_k^0 \varphi_k(x)\,, \tag{2.6}$$

where $f_k = \langle f(x), \varphi_k(x)\rangle$, $u_k^0 = \langle u^0, \varphi_k(x)\rangle$, $k = 1, 2, \cdots$, and $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\Omega)$. For any given source term $f(x)$ and initial function $u^0(x)$, in solving problem (1.1) we can formally define two linear operators:

$$K_1 : (f, u^0) \mapsto u(x, T_1)\,, \tag{2.7}$$

$$K_2 : (f, u^0) \mapsto u(x, T_2)\,. \tag{2.8}$$

Similarly, we can define four linear operators for the problems (2.1) and (2.2), respectively — viz.

$$K_{1,i} : f \mapsto u_1(x, T_i)\,, \quad i = 1, 2\,, \tag{2.9}$$

$$K_{2,i} : u^0 \mapsto u_2(x, T_i)\,, \quad i = 1, 2\,. \tag{2.10}$$

From the solution expressions (2.5) and (2.6), we obtain the equations

$$\big(K_{1,i}(f)\big)(x) = \sum_{k=1}^{+\infty} f_k \frac{1 - E_{\alpha,1}(-\lambda_k T_i^\alpha)}{\lambda_k} \varphi_k(x)\,, \quad i = 1, 2\,, \tag{2.11}$$

$$\big(K_{2,i}(u^0)\big)(x) = \sum_{k=1}^{\infty} E_{\alpha,1}(-\lambda_k T_i^\alpha) u_k^0 \varphi_k(x)\,, \quad i = 1, 2\,; \tag{2.12}$$

and to identify $u^0(x)$ and $f(x)$ simultaneously we then have operator equations

$$\big(K_1(f, u^0)\big)(x) = \big(K_{1,1}(f)\big)(x) + \big(K_{2,1}(u^0)\big)(x) = g_1(x)\,, \tag{2.13}$$

$$\big(K_2(f,u^0)\big)(x)=\big(K_{1,2}(f)\big)(x)+\big(K_{2,2}(u^0)\big)(x)=g_2(x)\,, \tag{2.14}$$

which can be written in the matrix form

$$B\begin{pmatrix} f \\ u^0 \end{pmatrix}=\begin{pmatrix} g_1 \\ g_2 \end{pmatrix},\ \text{where}\ B=\begin{pmatrix} K_{1,1} & K_{2,1} \\ K_{1,2} & K_{2,2} \end{pmatrix}. \tag{2.15}$$

3. Ill-Posedness and Conditional Stability for the Simultaneous Problem

We define the Hilbert space

$$D\big((L)^\theta\big)=\left\{\chi\in L^2(\Omega):\ (\sum_{k=1}^{\infty}\lambda_k^{2\theta}|\langle\chi,\varphi_k\rangle|^2)^{\frac{1}{2}}<\infty\right\}, \tag{3.1}$$

where

$$\|\chi\|_{D((L)^\theta)}=(\sum_{k=1}^{\infty}\lambda_k^{2\theta}|\langle\chi,\varphi_k\rangle|^2)^{\frac{1}{2}}. \tag{3.2}$$

Theorem 3.1. *Suppose $f(x)$ and $u^0(x)\in(D(L)^\theta)$ with nonnegative θ satisfy the a priori condition*

$$\max\left\{\|f(x)\|_{D\left((L)^\theta\right)},\|u^0(x)\|_{D\left((L)^\theta\right)}\right\}\le E\,. \tag{3.3}$$

Then $u(\cdot,t)\in D\left((L)^{\theta+1}\right)$, and $\|u(\cdot,t)\|_{D((L)^{\theta+1})}\le(1+C/t^\alpha)E$ for any $t>0$.

Proof. From Eq. (2.3), the solution to the problem (1.1) can be expressed as

$$u(x,t)=\sum_{k=1}^{\infty}\left(u_k^0E_{\alpha,1}(-\lambda_kt^\alpha)+f_k\frac{1-E_{\alpha,1}(-\lambda_kt^\alpha)}{\lambda_k}\right)\varphi_k(x)\,.$$

From Lemma 2.1,

$$\begin{aligned}\|u(\cdot,t)\|_{D((L)^{\theta+1})}=&\left(\sum_{k=1}^{\infty}\lambda_k^{2\theta+2}\left(u_k^0E_{\alpha,1}(-\lambda_kt^\alpha)+f_k\frac{1-E_{\alpha,1}(-\lambda_kt^\alpha)}{\lambda_k}\right)^2\right)^{\frac{1}{2}}\\ \le&\left(\sum_{k=1}^{\infty}\lambda_k^{2\theta+2}(u_k^0E_{\alpha,1}(-\lambda_kt^\alpha))^2\right)^{\frac{1}{2}}+\left(\sum_{k=1}^{\infty}\lambda_k^{2\theta+2}(f_k\frac{1-E_{\alpha,1}(-\lambda_kt^\alpha)}{\lambda_k})^2\right)^{\frac{1}{2}}\\ \le&\left(\sum_{k=1}^{\infty}\lambda_k^{2\theta}(u_k^0)^2\lambda_k^2(\frac{C}{\lambda_kt^\alpha})^2\right)^{\frac{1}{2}}+\left(\sum_{k=1}^{\infty}\lambda_k^{2\theta}(f_k)^2\lambda_k^2\frac{1}{(\lambda_k)^2}\right)^{\frac{1}{2}}\\ \le&\frac{C}{t^\alpha}\|u^0\|_{D((L)^\theta)}+\|f\|_{D((L)^\theta)}\,,\end{aligned}$$

which completes the proof. □

From Theorem 3.1, we obtain $u(x,T_1)$, $u(x,T_2)\in D((L)^1)(\Omega)$ if f, $u^0\in L^2(\Omega)$. Since $D((L)^1)$ is compactly imbedded into $L^2(\Omega)$, the operator $B:(L^2(\Omega))^2\to(L^2(\Omega))^2$ is compact. The problem of simultaneously determining the initial data and the source term is therefore ill-posed.

Next, we consider a conditional stability for the simultaneous reconstruction problem.

Theorem 3.2. *Denote by $g_i = u(\cdot, T_i)$ for $i = 1,2$. If f, $u^0 \in D((L)^\theta)$ satisfy the a priori bound condition* (3.3) *for some positive θ, then*

$$\|f\| \le CE^{\frac{1}{\theta+1}}\left(\|g_1\|+\left(\frac{T_2}{T_1}\right)^{\alpha}\|g_2\|\right)^{\frac{\theta}{\theta+1}}, \ \ \|u^0\| \le CE^{\frac{1}{\theta+1}}\left(\frac{1}{1-E_{\alpha,1}(-\lambda_1 T_1^\alpha)}\|g_1\|+\|g_2\|\right)^{\frac{\theta}{\theta+1}}.$$

A proof is given in Appendix A.

Remark 3.1. The conditional stability result in Theorem 3.2 partly implies the uniqueness of the simultaneous identification problem if f, $u^0 \in D((L)^\theta)$ for some positive θ.
A more complete uniqueness result under the weaker assumption f, $u^0 \in L^2(\Omega)$ can also be obtained via (A.1) in Appendix A. Let $g_i(x) = 0$ for $i = 1,2$, then $g_{i,k} = \langle g_i(x), \phi_k(x)\rangle = 0$. Since the denominator $e^1_{1,k}e^2_{2,k} - e^1_{2,k}e^2_{1,k} \neq 0$ from (A.2), we conclude $f_k = u_{0,k} = 0$ and obtain the uniqueness of the inverse problem.

4. Tikhonov Regularisation and Convergence Analysis

From Theorem 3.1, we know that the simultaneous identification problem is ill-posed due to compactness of the operator B, so regularisation is necessary to recover the source $f(x)$ and initial value $u^0(x)$. We adopt the standard Tikhonov regularisation method to solve the ill-posed problem, which minimises the functional $J(f,u^0)$ — i.e.

$$\min J(f,u^0) = \frac{1}{2}\left\|B\begin{pmatrix} f \\ u^0 \end{pmatrix} - \begin{pmatrix} g_1^\delta \\ g_2^\delta \end{pmatrix}\right\|^2 + \frac{1}{2}\left\|\begin{pmatrix} \sqrt{\mu} & 0 \\ 0 & \sqrt{\gamma} \end{pmatrix}\begin{pmatrix} f \\ u^0 \end{pmatrix}\right\|^2, \tag{4.1}$$

where μ, $\gamma > 0$ are the regularisation parameters. From Theorem 2.12 in Ref. [2], there are unique minimisers:

$$\begin{pmatrix} f^\delta_{\mu,\gamma} \\ u^{0,\delta}_{\mu,\gamma} \end{pmatrix} = \left(B^*B + \begin{pmatrix} \mu I & 0 \\ 0 & \gamma I \end{pmatrix}\right)^{-1} \cdot B^* \begin{pmatrix} g_1^\delta \\ g_2^\delta \end{pmatrix}, \tag{4.2}$$

where B^* is the adjoint of the operator B. (We will omit the superscript δ if there is no noise.) The two regularisation parameters μ and γ can be chosen independently, but to simplify the analysis here we set $\mu = \gamma$ and write $(f^\delta_\mu, u^{0,\delta}_\mu) := (f^\delta_{\mu,\gamma}, u^{0,\delta}_{\mu,\gamma})$ or (f_μ, u^0_μ) for the noise-free case.

4.1. Convergence rate for an *a priori* parameter choice rule

We write

$$e^j_{1,k} = \frac{1}{\lambda_k}(1 - E_{\alpha,1}(-\lambda_k T_j^\alpha)) \text{ and } e^j_{2,k} = E_{\alpha,1}(-\lambda_k T_j^\alpha), \quad j = 1,2, \ k = 1,\cdots,\infty, \tag{4.3}$$

to address the following technique lemma.

Lemma 4.1. *Let* $\omega_1(k)=\frac{(e^1_{1,k}e^2_{2,k}-e^2_{1,k}e^1_{2,k})^2}{e^1_{1,k}e^1_{2,k}+e^2_{1,k}e^2_{2,k}}$, $\omega_2(k)=\frac{(e^1_{1,k}e^2_{2,k}-e^2_{1,k}e^1_{2,k})^2}{(e^2_{1,k})^2+(e^2_{2,k})^2}$, $\omega_3(k)=\frac{(e^1_{1,k}e^2_{2,k}-e^2_{1,k}e^1_{2,k})^2}{(e^1_{1,k})^2+(e^2_{1,k})^2}$. *Then*

$$\lim_{k\to+\infty}\lambda_k^2\omega_1(k)=\frac{(T_1^\alpha-T_2^\alpha)^2}{\Gamma(1-\alpha)T_1^\alpha T_2^\alpha(T_1^\alpha+T_2^\alpha)},$$
$$\lim_{k\to+\infty}\lambda_k^2\omega_2(k)=\frac{T_2^{2\alpha}}{1+\Gamma^2(1-\alpha)T_2^{2\alpha}}\left(\frac{1}{T_2^\alpha}-\frac{1}{T_1^\alpha}\right)^2,$$
$$\lim_{k\to+\infty}\lambda_k^2\omega_3(k)=\frac{1}{\Gamma^2(1-\alpha)}\left(\frac{1}{T_2^\alpha}-\frac{1}{T_1^\alpha}\right)^2.$$

A proof is given Appendix B.

The convergence rate by an *a priori* parameter selection rule is then as follows.

Theorem 4.1. *Suppose the smoothness assumption* (3.3) *holds.*
(a) *If* $0<\theta<2$, *one can choose* $\gamma=\mu=(\delta/E)^{\frac{2}{\theta+1}}$ *such that*

$$\left\|u_\mu^{0,\delta}(x)-u^0(x)\right\|+\left\|f_\mu^\delta(x)-f(x)\right\|\le CE^{\frac{1}{1+\theta}}\delta^{\frac{\theta}{1+\theta}}.$$

(b) *If* $\theta\ge 2$, *one can choose* $\gamma=\mu=(\delta/E)^{\frac{2}{3}}$ *such that*

$$\left\|u_\mu^{0,\delta}(x)-u^0(x)\right\|+\left\|f_\mu^\delta(x)-f(x)\right\|\le CE^{\frac{1}{3}}\delta^{\frac{2}{3}}.$$

A proof is given Appendix C.

4.2. Convergence rate by an *a posteriori* parameter choice rule

We now provide the convergence rate for an *a posterior* regularisation parameter choice strategy, based on the following Morozov discrepancy principle:

$$\tau_1\delta\le\left\|(K_1(f_\mu^\delta,u_\mu^{0,\delta}))(x)-g_1^\delta(x)\right\|+\left\|(K_2(f_\mu^\delta,u_\mu^{0,\delta}))(x)-g_2^\delta(x)\right\|\le\tau_2\delta\,, \tag{4.4}$$

where τ_i, $i=1,2$ are two given constants specified later. First, we consider the existence of the parameter that satisfies inequality (4.4) if $\|g_1^\delta(x)\|+\|g_2^\delta(x)\|>\tau_2\delta>0$.

Lemma 4.2. *If* $\rho(\mu)=\|K_1(f_\mu^\delta,u_\mu^{0,\delta})(x)-g_1^\delta(x)\|+\|K_2(f_\mu^\delta,u_\mu^{0,\delta})(x)-g_2^\delta(x)\|$, *then* $\rho(\mu)$ *is a continuous function such that* $\lim_{\mu\to0}\rho(\mu)=0$, $\lim_{\mu\to+\infty}\rho(\mu)=\|g_1^\delta(x)\|+\|g_2^\delta(x)\|$.

Proof. By computation and the expressions (C.2), we can reformulate $\rho(\mu)$ as

$$\rho(\mu)=\left(\sum_{k=1}^\infty\left(\frac{(-\mu(e^2_{2,k})^2+\mu(e^2_{1,k})^2+\mu^2)g^\delta_{1,k}+(\mu e^2_{1,k}e^1_{1,k}+\mu e^1_{2,k}e^2_{2,k})g^\delta_{2,k}}{(e^1_{1,k}e^2_{2,k}-e^2_{1,k}e^1_{2,k})^2+\mu((e^1_{2,k})^2+(e^2_{2,k})^2)+\mu((e^1_{1,k})^2+(e^2_{1,k})^2)+\mu^2}\right)^2\right)^{\frac{1}{2}}$$

$$+\left(\sum_{i=1}^{\infty}\left(\frac{\mu(e^1_{1,i}e^2_{1,i}+e^1_{2,i}e^2_{2,i})g^\delta_{1,i}-\mu((e^1_{1,i})^2+(e^1_{2,i})^2+\mu)g^\delta_{2,i}}{(e^1_{1,i}e^2_{2,i}-e^2_{1,i}e^1_{2,i})^2+\mu((e^1_{2,i})^2+(e^2_{2,i})^2+(e^1_{1,i})^2+(e^2_{1,i})^2)+\mu^2}\right)^2\right)^{\frac{1}{2}}, \tag{4.5}$$

whence the desired result. □

We next provide some technique Lemmas.

Lemma 4.3. *If* $F(t)=\frac{E_{\alpha,1}(-T_1^\alpha t)}{E_{\alpha,1}(-T_2^\alpha t)}$, $t>0$, *then* $F(t)$ *is a nondecreasing function for any* $t>0$.

Proof. Taking the derivative of $F(t)$, we obrain

$$F'(t)=\frac{T_2^\alpha E'_{\alpha,1}(-T_2^\alpha t)E_{\alpha,1}(-T_1^\alpha t)-T_1^\alpha E'_{\alpha,1}(-T_1^\alpha t)E_{\alpha,1}(-T_2^\alpha t)}{E^2_{\alpha,1}(-T_2^\alpha t)}.$$

From Lemma 2.1, $E'_{\alpha,1}(-T_2^\alpha t)>E'_{\alpha,1}(-T_1^\alpha t)$, which implies $F'(t)\geq 0$. □

We introduce a pair of auxiliary variables $(f^\delta,u^{0,\delta})$, corresponding to the regularisation parameters $\mu=\gamma=0$ — viz.

$$\begin{cases} f^\delta=\sum\limits_{k=1}^{\infty}f_k^\delta\varphi_k(x)\,, \\ u^{0,\delta}=\sum\limits_{k=1}^{\infty}u_k^{0,\delta}\varphi_k(x)\,, \end{cases} \tag{4.6}$$

where

$$f_k^\delta=\frac{g^\delta_{1,k}e^2_{2,k}-g^\delta_{2,k}e^1_{2,k}}{e^1_{1,k}e^2_{2,k}-e^2_{1,k}e^1_{2,k}},\quad k=1,\cdots,\infty\,,$$
$$u^\delta_{0,k}=\frac{g^\delta_{2,k}e^1_{1,k}-g^\delta_{1,k}e^2_{1,k}}{e^1_{1,k}e^2_{2,k}-e^2_{1,k}e^1_{2,k}},\quad k=1,\cdots,\infty\,,$$

and $e^1_{1,k}$, $e^2_{1,k}$, $e^1_{2,k}$, $e^2_{2,k}$ are defined by expression (4.3). It is notable that the terms $(f^\delta,u^{0,\delta})$ are not in L^2 space, and the summation of (4.6) is to be understood in the distribution sense. Moreover,

$$K_{1,j}f^\delta=\sum_{k=1}^{\infty}f_k^\delta K_{1,j}\varphi_k(x)\,,\qquad K_{2,j}u^{0,\delta}=\sum_{k=1}^{\infty}u_k^{0,\delta}K_{2,j}\varphi_k(x)\,,$$

which are both in $L^2(\Omega)$ (as proven in Appendix D). For convenience, we introduce the notation $\Delta f=f^\delta-f$, $\Delta u^0=u^{0,\delta}-u^0$, $\Delta g_1=g_1^\delta-g_1$, $\Delta g_2=g_2^\delta-g_2$ in the following result.

Lemma 4.4. *If* $\|g_1^\delta-g_1\|\leq\delta$, $\|g_2^\delta-g_2\|\leq\delta$, *we have the inequality*

$$\Big\|(K_{1,j}\Delta f)(x)\Big\|_{L^2(\Omega)}\leq C\delta,\qquad \Big\|(K_{2,j}\Delta u^0)(x)\Big\|_{L^2(\Omega)}\leq C\delta\,,\qquad j=1,2.$$

If the inequalities (4.4) *are satisfied, then*

$$\left\|K_{1,j}(f_\mu^\delta - f^\delta)\right\| \le C\delta\,, \qquad \left\|K_{2,j}(u_\mu^{0,\delta} - u^{0,\delta})\right\| \le C\delta\,, \qquad j=1,2\,.$$

A proof is given in Appendix D.

Lemma 4.5. *Suppose inequality* (3.3) *holds for some positive θ, and μ satisfies the Morozov discrepancy principle* (4.4)*. Then*

(a) $\|K_{1,j}(f_\mu - f)\| \le C\delta, \|K_{2,j}(u_\mu^0 - u^0)\| \le C\delta\ ,\ j=1,2\,;$

(b) $u_\mu^0 - u^0,\ f_\mu - f \in D((L)^\theta)\,,\ \|f_\mu - f\|_{D((L)^\theta)} \le CE,\ \|u_\mu^0 - u^0\|_{D((L)^\theta)} \le CE,$

$$\left\|f_\mu - f\right\|_{L^2(\Omega)} \le C\delta^{\frac{\theta}{\theta+1}}\,, \qquad \left\|u_\mu^0 - u^0\right\|_{L^2(\Omega)} \le C\delta^{\frac{\theta}{\theta+1}}\,. \tag{4.7}$$

A proof is given in Appendix E.

We now consider the convergence rate for our *a posteriori* parameter selection rule.

Theorem 4.2. *Suppose inequality* (3.3) *holds for some positive θ, and μ satisfies the Morozov discrepancy principle* (4.4) *for some $\tau_i > 3$, $i=1,2$ and $\|g_1^\delta(x)\| + \|g_2^\delta(x)\| > \tau_2\delta > 0$.*

(a) *If $0<\theta<1$, then*

$$\left\|u_\mu^{0,\delta}(x) - u^0(x)\right\| + \left\|f_\mu^\delta(x) - f(x)\right\| \le C\delta^{\frac{\theta}{1+\theta}}\,.$$

(b) *If $\theta \ge 1$, then*

$$\left\|u_\mu^{0,\delta}(x) - u^0(x)\right\| + \left\|f_\mu^\delta(x) - f(x)\right\| \le C\sqrt{\delta}\,.$$

A proof is given in Appendix F.

5. Numerical Calculations

5.1. Discretisation method

We use a standard finite element discretisation for the spatial variable and a finite difference discretisation for the time-fractional derivative. Thus let the time interval $[0, T_2]$ be partitioned into N_2 equal subintervals with nodes $0 = t_0 < t_1 < \cdots < t_{N_2-1} < t_{N_2} = T_2$, where $t_k = k\tau$, $t_{N_1} = T_1$, $\tau = T_2/N_2$. Then the time-fractional derivative $\partial^\alpha u(x,t)/\partial t^\alpha$ at t_k is approximated by

$$\begin{aligned}\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\Big|_{t=t_k} &= \frac{1}{\Gamma(1-\alpha)}\int_0^{t_k}(t_k-\eta)^{-\alpha}\frac{\partial u(x,\eta)}{\partial\eta}\mathrm{d}\eta \\ &= \frac{1}{\Gamma(1-\alpha)}\sum_{l=1}^{k}\int_{t_{l-1}}^{t_l}(t_k-\eta)^{-\alpha}\frac{\partial u(x,\eta)}{\partial\eta}\mathrm{d}\eta\end{aligned}$$

$$\approx \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{l=1}^{k}(u(x,t_l)-u(x,t_{l-1}))\big((k+1-l)^{1-\alpha}-(k-l)^{1-\alpha}\big)$$
$$=\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{l=1}^{k}\omega_l\left(u(x,t_{k+1-l})-u(x,t_{k-l})\right),$$

where $\omega_l = l^{1-\alpha}-(l-1)^{1-\alpha}$, $l=1,2,\cdots,k$ and $k=1,2,\cdots,N_2$, so the discrete time-fractional difference form is

$$Du_{h,t}^{\alpha,k}=\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{l=1}^{k}\omega_l\left(u_h^{k+1-l}-u_h^{k-l}\right).$$

The spatial discretisation chosen is a continuous piecewise linear finite element space V_h over a quasi-uniform triangulation T_h — viz.

$$V_h=\left\{\, v: v\in C_0(\Omega),\ v\big|_{\Delta_h}\in P_1(\Delta_h),\ \forall\ \Delta_h\in\ T_h\,\right\},$$

with a standard basis $\{\psi_i\}_0^{N_0}$. On defining a bilinear form

$$a(u,\chi)=\int_\Omega\left(\sum_{i,j=1}^{d}a_{i,j}\frac{\partial u(x)}{\partial x_i}\cdot\chi_{x_j}+c(x)u\cdot\chi\right)\mathrm{d}x,$$

the full discrete scheme is thus

$$\langle Du_{h,t}^{\alpha,k},\psi\rangle+a(u_h^k,\psi)=\langle f_h,\psi\rangle,\ u_h^0=u^0(x) \tag{5.1}$$

for any $\psi\in V_h$. Eq. (5.1) defines the discrete linear operators $K_{i,h}:(f,u^0)\mapsto u^{N_i}, i=1,2$; and for the system of equations (5.1) we have the discrete optimisation problem

$$\begin{aligned}&\min_{f_h,u_h^0\in V_h} J(f_h,u_h^0)\\ =&\frac{1}{2}\int_\Omega\left(\sum_{i=0}^{N_0}(f_iu_{1,h,i}^{N_1}+u_i^0u_{2,h,i}^{N_1})-g_{1,h}\right)^2\mathrm{d}x+\frac{1}{2}\int_\Omega\left(\sum_{i=0}^{N_0}(f_iu_{1,h,i}^{N_2}+u_i^0u_{2,h,i}^{N_2})-g_{2,h}\right)^2\mathrm{d}x\\ &+\frac{\mu}{2}\int_\Omega\left(\sum_{i=0}^{N_0}f_i\psi_i(x)\right)^2\mathrm{d}x+\frac{\gamma}{2}\int_\Omega\left(\sum_{i=0}^{N_0}u_i^0\psi_i(x)\right)^2\mathrm{d}x,\end{aligned} \tag{5.2}$$

where $u_{1,h,i}^k$, $k=1,2,\cdots,N_2$ is the solution of

$$\langle Du_{1,h,t}^{\alpha,k},\psi_j\rangle+a(u_{1,h,i}^k,\psi_j)=\langle\psi_i,\psi_j\rangle\ \ \forall j,\qquad u_{1,h,i}^0=0,$$

and $u_{2,h,i}^k$, $k=1,2,\cdots,N_2$ the solution of

$$\langle Du_{2,h,t}^{\alpha,k},\psi_j\rangle+a(u_{2,h,i}^k,\psi_j)=0\ \ \forall j,\qquad u_{2,h,i}^0=\psi_i.$$

The solution of discrete optimisation problem $\Phi = [f_0, \cdots, f_{N_0}, u_0^0, \cdots, u_{N_0}^0]^T$ thus satisfies the linear algebraic system

$$A\Phi = F\ , \tag{5.3}$$

with $A = [A_{11}A_{12}; A_{21}, A_{22}]$ and $F = [F_1; F_2]$ where

$$A_{11} = \left[\iint_\Omega \left(u_{1,h,i}^{N_1}(x)u_{1,h,j}^{N_1}(x) + u_{1,h,i}^{N_2}(x)u_{1,h,j}^{N_2}(x) + \mu\psi_i\psi_j\right)dx\right]_{(N_0+1)\times(N_0+1)},$$

$$A_{12} = \left[\iint_\Omega \left(u_{1,h,i}^{N_1}(x)u_{2,h,j}^{N_1}(x) + u_{1,h,i}^{N_2}(x)u_{2,h,j}^{N_2}(x)\right)dx\right]_{(N_0+1)\times(N_0+1)},$$

$$A_{21} = \left[\iint_\Omega \left(u_{2,h,i}^{N_1}(x)u_{1,h,j}^{N_1}(x) + u_{2,h,i}^{N_2}(x)u_{1,h,j}^{N_2}(x)\right)dx\right]_{(N_0+1)\times(N_0+1)},$$

$$A_{22} = \left[\iint_\Omega \left(u_{2,h,i}^{N_1}(x)u_{2,h,j}^{N_1}(x) + u_{2,h,i}^{N_2}(x)u_{2,h,j}^{N_2}(x) + \gamma\psi_i\psi_j\right)dx\right]_{(N_0+1)\times(N_0+1)},$$

$$F_1 = \left[\iint_\Omega g_{1,h}(x)u_{1,h,i}^{N_1}(x) + g_{2,h}(x)u_{1,h,i}^{N_2}(x)\mathrm{d}x\right]_{(N_0+1)\times 1},$$

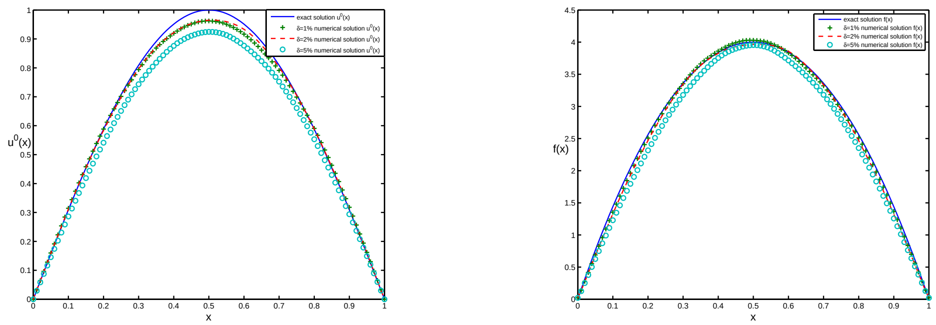
$$F_2 = \left[\iint_\Omega g_{1,h}(x)u_{2,h,i}^{N_1}(x) + g_{2,h}(x)u_{2,h,i}^{N_2}(x)\mathrm{d}x\right]_{(N_0+1)\times 1}.$$

From the solution $\Phi^* = [f_0^*, f_1^*, \cdots, f_{N_0}^*, u_0^{0,*}, u_1^{0,*}, \cdots, u_{N_0}^{0,*}]^T$ of Eq. (5.3), we then obtain the approximate solutions

$$f_h = \sum_{i=0}^{N_0} f_i^*\psi_i(x) \quad \text{and} \quad u_h^0 = \sum_{i=0}^{N_0} u_i^{0,*}\psi_i(x)\,. \tag{5.4}$$

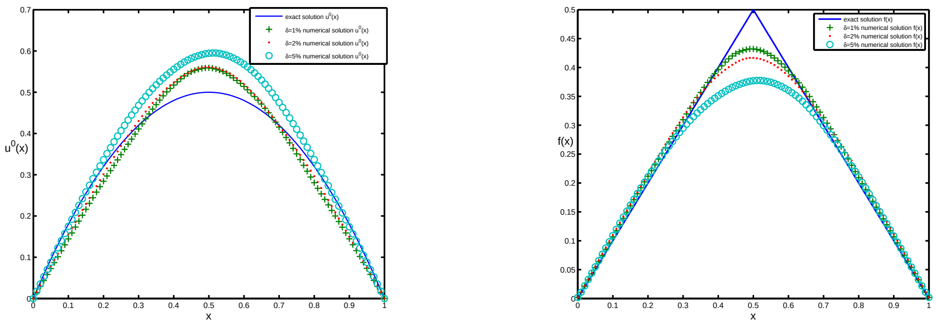
5.2. Numerical examples

We present five numerical examples to illustrate the efficiency of the proposed method. The first two are for one-dimensional problem. Let $\Omega = [0,1]$ be divided into 100 equal subintervals, such that $\tau = 1/100$ and $Lu(x,t) = -\mathrm{d}/\mathrm{d}x(a(x)\mathrm{d}u/\mathrm{d}x) + c(x)u$. The last three examples are two-dimensional problems, where $\Omega = [0,1]\times[0,1]$ is partitioned into 1024 triangular elements, τ is 1/80 and $Lu(x,t) = -\nabla(a(x,y)\nabla u) + c(x,y)u$. The observation times are chosen to be $T_1 = 1/2$ and $T_2 = 1$. The noisy data are defined by $g_i^\delta(x_j) = u_h^{N_i}(x_j) * (1+\delta\xi), i = 1,2$, where δ is the noise level and ξ is a uniform distributed random variable on $[-1,1]$. In order to obtain good regularisation solutions, some proper regularisation parameters are necessary [23]. In our computation, we first fix the ratio of the two regularisation parameters μ and γ, and the discrepancy principle is then applied via a decreasing geometric sequence, in order to choose the regularisation parameters [3,4,19].



(a) inversion of $u^0(x)$ (b) inversion of $f(x)$

Figure 1: Numerical solution for Example 5.1.



(a) inversion of $u^0(x)$ (b) inversion of $f(x)$

Figure 2: Numerical solution for Example 5.2.

Example 5.1. Consider $\alpha = 0.9$, $a(x) = 1$ and $c(x) = 0$. Let the exact source term and initial value for the problem (1.1) be $16x(1-x)$ and $\sin(\pi x)$, respectively. Numerical results for the relative noise levels $1\%, 2\%, 5\%$ are shown in Fig. 1.

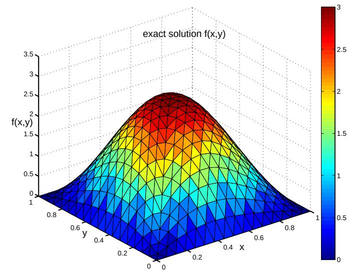
Example 5.2. For problem (1.1), consider the respective exact source term and initial value

$$f^{\dagger} = \begin{cases} x, & x \in [0, \frac{1}{2}) \\ (1-x), & x \in [\frac{1}{2}, 1], \end{cases} \qquad u^{0,\dagger} = 2x(1-x).$$

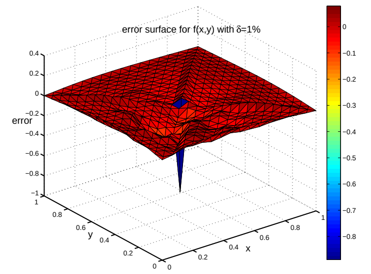
For $a(x) = 1$, $c(x) = 0$ and $\alpha = 0.9$, the results for the relative noise levels 1%, 2% and 5% are shown in Fig. 2.

Example 5.3. Consider the exact source term and initial data $e^{2-x-y}\sin(\pi x)\sin(\pi y)$ and $xy(1-x)(1-y)e^{4-x-y}$, respectively. For $\alpha = 0.8$, $a(x,y) = 1$ and $c(x,y) = 0$, the results are shown in Figs. 3 and 4 for the relative noise levels 1%, 2% and 5%.

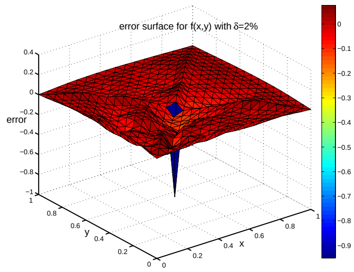
Example 5.4. Consider the exact source term and initial value $16xy(1-x)(1-y)$ and $\sin(\pi x)\sin(\pi y)$, respectively. For $\alpha = 0.8$, $a(x,y) = 0.01(x+y)$ and $c(x,y) = 0.001$, the results are shown in Figs. 5 and 6 for the relative noise levels 1%, 2% and 5%.



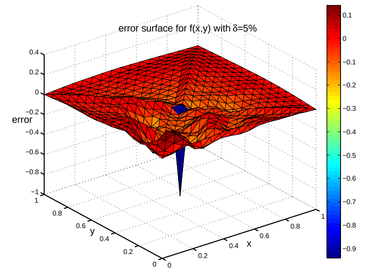
(a) exact source term



(b) error to the source term with $\delta = 1\%$

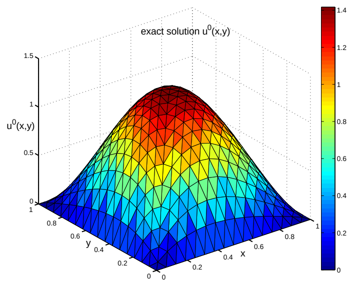


(c) error to the source term with $\delta = 2\%$

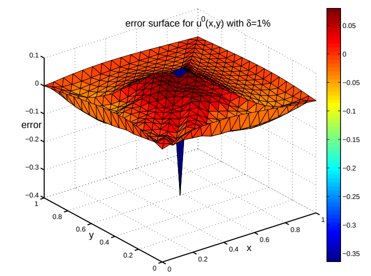


(d) error to the source term with $\delta = 5\%$

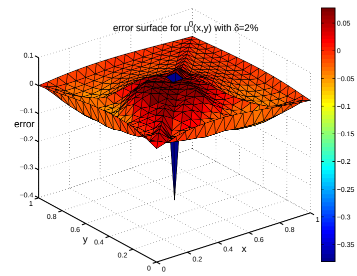
Figure 3: Numerical results for the source term in Example 5.3.



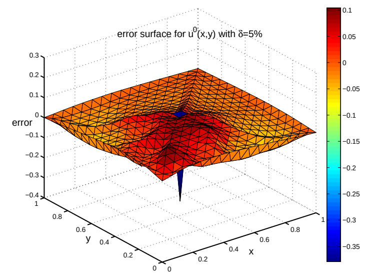
(a) exact initial data



(b) error to the initial data with $\delta = 1\%$

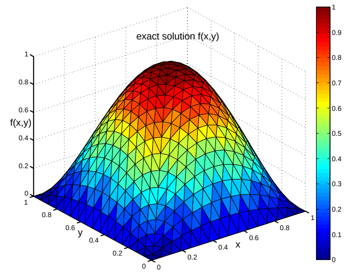


(c) error to the initial data with $\delta = 2\%$

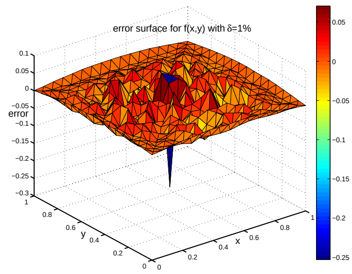


(d) error to the initial data with $\delta = 5\%$

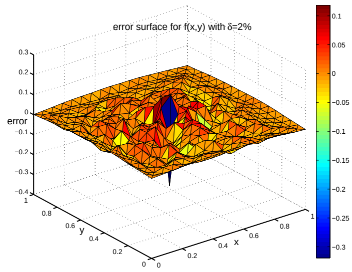
Figure 4: Numerical results for the initial data in Example 5.3.



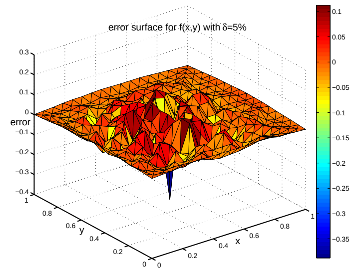
(a) exact solution



(b) error with $\delta = 1\%$

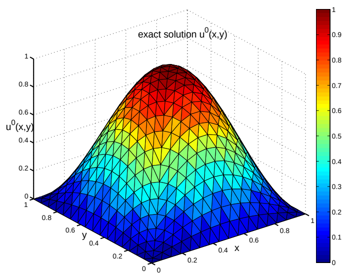


(c) error with $\delta = 2\%$

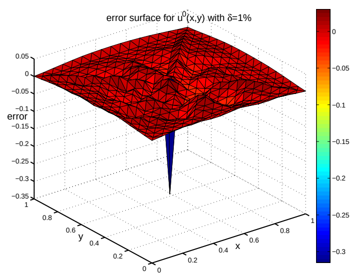


(d) error with $\delta = 5\%$

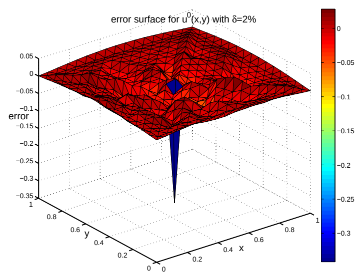
Figure 5: Numerical results for the source term in Example 5.4.



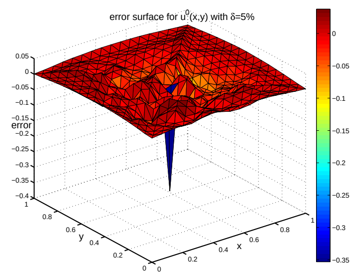
(a) exact solution



(b) error with $\delta = 1\%$

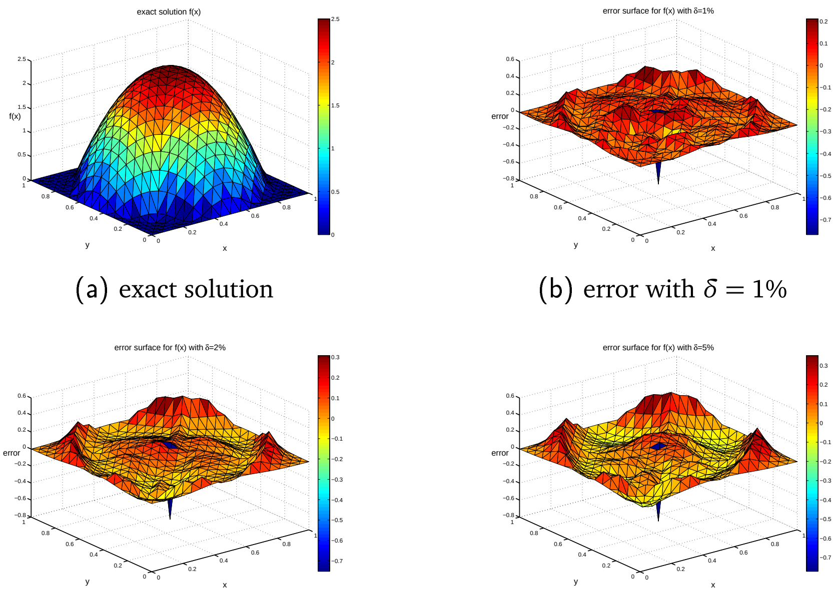


(c) error with $\delta = 2\%$



(d) error with $\delta = 5\%$

Figure 6: Numerical results for the initial data in Example 5.4.



(a) exact solution (b) error with $\delta = 1\%$

(c) error with $\delta = 2\%$ (d) error with $\delta = 5\%$

Figure 7: Numerical results for the source term in Example 5.5.

Example 5.5. Consider the exact source term and initial value for the problem (1.1):

$$f^\dagger = \begin{cases} 10\left(\frac{1}{4} - (x-\frac{1}{2})^2 - (y-\frac{1}{2})^2\right), & \left(x-\frac{1}{2})^2 + (y-\frac{1}{2}\right)^2 \leq \frac{1}{4}, \\ 0, \ else, \end{cases}$$

$$u^{0,\dagger} = \begin{cases} 10\sin\left(\frac{1}{4} - (x-\frac{1}{2})^2 - (y-\frac{1}{2})^2\right), & \left(x-\frac{1}{2})^2 + (y-\frac{1}{2}\right)^2 \leq \frac{1}{4}, \\ 0, \ else. \end{cases}$$

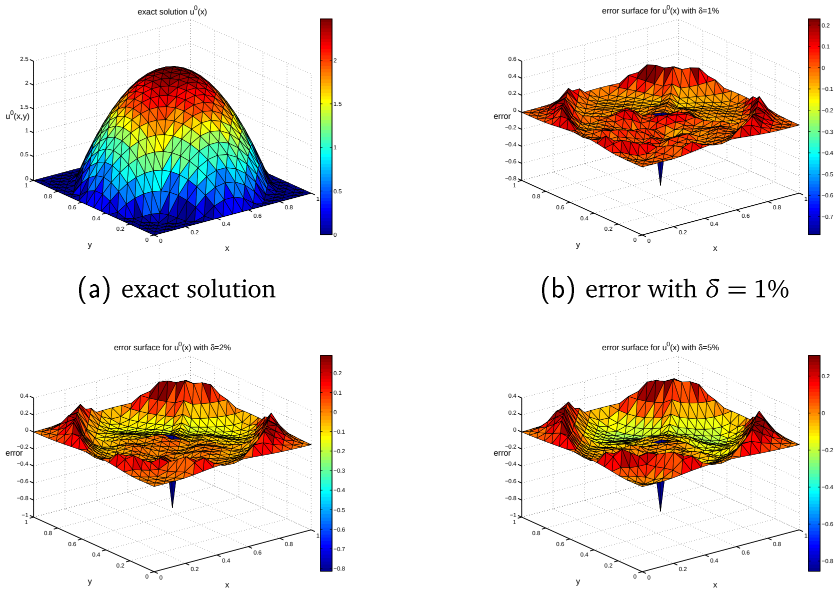
For $\alpha = 0.6$, $a(x,y) = 1$ and $c(x,y) = 0$, the results are shown in Figs. 7 and 8 for the relative noise levels 1%, 2% and 5%.

6. Conclusions

The inverse problem investigated in this paper involves simultaneously identifying the time-independent source function and initial data for a time-fractional diffusion equation. By Tikhonov regularisation, this inverse problem is transformed into an optimisation problem. The conditional stability is provided. *A priori* and *a posterior* parameter selection rules and corresponding convergence rates are found, and several numerical examples demonstrate the efficiency of the proposed algorithm.

Acknowledgments

The authors would like to thank the anonymous referees for useful comments. The work of J. Z. Yang is partly supported by the National Science Foundation of China grants



(a) exact solution (b) error with $\delta = 1\%$

(c) error with $\delta = 2\%$ (d) error with $\delta = 5\%$

Figure 8: Numerical results for the initial data in Example 5.5.

No. 11171305 and No. 91230203, and the work of X. Lu is partly supported by the National Science Foundation of China grants No. 91230108 and No. 11471253.

A. Proof of Theorem 3.2

We have $g_{i,k} = (g_i(x), \varphi_k)$, $i = 1,2$, $k = 1,2,\cdots$, and $t = T_1, T_2$. From the expression (2.3),

$$g_i(x) = \sum_{k=1}^{\infty} E_{\alpha,1}(-\lambda_k T_i^{\alpha}) u_{0,k}\varphi_k(x) + \sum_{k=1}^{\infty} \frac{1}{\lambda_k}\big(1 - E_{\alpha,1}(-\lambda_k T_i^{\alpha})\big) f_k \varphi_k(x), \qquad i = 1,2.$$

By simple calculation,

$$\begin{cases} f_k = \dfrac{g_{1,k}e_{2,k}^2 - g_{2,k}e_{2,k}^1}{e_{1,k}^1 e_{2,k}^2 - e_{1,k}^2 e_{2,k}^1}, \\ u_{0,k} = \dfrac{g_{2,k}e_{1,k}^1 - g_{1,k}e_{1,k}^2}{e_{1,k}^1 e_{2,k}^2 - e_{1,k}^2 e_{2,k}^1}, \end{cases} \tag{A.1}$$

where $e_{1,k}^1$, $e_{1,k}^2$, $e_{2,k}^1$, $e_{2,k}^2$ are defined from expression (4.3). Using the Hölder inequality, we have

$$\|f\|^2 = \sum_{k=1}^{\infty}\left(\frac{g_{1,k}e_{2,k}^2 - g_{2,k}e_{2,k}^1}{e_{1,k}^1 e_{2,k}^2 - e_{1,k}^2 e_{2,k}^1}\right)^2 = \sum_{k=1}^{\infty} \frac{(g_{1,k} - g_{2,k}e_{2,k}^1/e_{2,k}^2)^{\frac{2}{\theta+1}}}{(e_{1,k}^1 - e_{1,k}^2 e_{2,k}^1/e_{2,k}^2)^2}\left(g_{1,k} - g_{2,k}e_{2,k}^1/e_{2,k}^2\right)^{\frac{2\theta}{\theta+1}}$$

$$\le \left(\sum_{k=1}^{\infty} \frac{(g_{1,k}-g_{2,k}e^1_{2,k}/e^2_{2,k})^2}{(e^1_{1,k}-e^2_{1,k}e^1_{2,k}/e^2_{2,k})^{2(\theta+1)}}\right)^{\frac{1}{\theta+1}} \left(\sum_{k=1}^{\infty}(g_{1,k}-g_{2,k}e^1_{2,k}/e^2_{2,k})^2\right)^{\frac{\theta}{\theta+1}}$$
$$=\left(\sum_{k=1}^{\infty} \frac{1}{(e^1_{1,k}-e^2_{1,k}e^1_{2,k}/e^2_{2,k})^{2\theta}}(f_k)^2\right)^{\frac{1}{\theta+1}} \left(\sum_{k=1}^{\infty}(g_{1,k}-g_{2,k}e^1_{2,k}/e^2_{2,k})^2\right)^{\frac{\theta}{\theta+1}}$$
$$\le \left(\sum_{k=1}^{\infty} \frac{1}{(e^1_{1,k}-e^2_{1,k}e^1_{2,k}/e^2_{2,k})^{2\theta}}(f_k)^2\right)^{\frac{1}{\theta+1}} \left(\Big(\sum_{k=1}^{\infty}(g_{1,k})^2\Big)^{\frac{1}{2}}+\Big(\sum_{k=1}^{\infty}(g_{2,k}e^1_{2,k}/e^2_{2,k}\Big)^2\Big)^{\frac{1}{2}}\right)^{\frac{2\theta}{\theta+1}};$$

and we also have

$$\left|e^1_{1,k}-e^2_{1,k}e^1_{2,k}/e^2_{2,k}\right| = \frac{1}{\lambda_k}\left(\frac{E_{\alpha,1}(-\lambda_k T_1^\alpha)}{E_{\alpha,1}(-\lambda_k T_2^\alpha)}-1\right).$$

It is not difficult to verify that $\frac{E_{\alpha,1}(-T_1^\alpha t)}{E_{\alpha,1}(-T_2^\alpha t)}$ is a nondecreasing function greater than 1 for any $t>0$, hence

$$\left|e^1_{1,k}-e^2_{1,k}e^1_{2,k}/e^2_{2,k}\right| = \frac{1}{\lambda_k}\left(\frac{E_{\alpha,1}(-\lambda_k T_1^\alpha)}{E_{\alpha,1}(-\lambda_k T_2^\alpha)}-1\right) \ge \frac{1}{\lambda_k}\left(\frac{E_{\alpha,1}(-\lambda_1 T_1^\alpha)}{E_{\alpha,1}(-\lambda_1 T_2^\alpha)}-1\right). \tag{A.2}$$

Moreover, $e^1_{2,k}/e^2_{2,k} = \frac{E_{\alpha,1}(-T_1^\alpha\lambda_k)}{E_{\alpha,1}(-T_2^\alpha\lambda_k)} \le \lim_{t\to\infty}\frac{E_{\alpha,1}(-T_1^\alpha t)}{E_{\alpha,1}(-T_2^\alpha t)} = (\frac{T_2}{T_1})^\alpha$ implies that

$$\|f\|^2 \le \left(\|g_1\|+(\frac{T_2}{T_1})^\alpha\|g_2\|\right)^{\frac{2\theta}{\theta+1}} \frac{1}{\left(\frac{E_{\alpha,1}(-\lambda_1 T_1^\alpha)}{E_{\alpha,1}(-\lambda_1 T_2^\alpha)}-1\right)^{\frac{2\theta}{\theta+1}}}\left(\|f\|_{(D(L)^\theta)}\right)^{\frac{2}{\theta+1}}.$$

By the same argument,

$$\|u^0\|^2 = \sum_{k=1}^{\infty}(\frac{g_{2,k}e^1_{1,k}-g_{1,k}e^2_{1,k}}{e^1_{1,k}e^2_{2,k}-e^2_{1,k}e^1_{2,k}})^2 = \sum_{k=1}^{\infty}\frac{(g_{2,k}-g_{1,k}e^2_{1,k}/e^1_{1,k})^{\frac{2}{\theta+1}}}{(e^2_{2,k}-e^1_{2,k}e^2_{1,k}/e^1_{1,k})^2}(g_{2,k}-g_{1,k}e^2_{1,k}/e^1_{1,k})^{\frac{2\theta}{\theta+1}}$$
$$\le \left(\sum_{k=1}^{\infty}\frac{(g_{2,k}-g_{1,k}e^2_{1,k}/e^1_{1,k})^2}{(e^2_{2,k}-e^1_{2,k}e^2_{1,k}/e^1_{1,k})^{2(\theta+1)}}\right)^{\frac{1}{\theta+1}}\left(\sum_{k=1}^{\infty}(g_{2,k}-g_{1,k}e^2_{1,k}/e^1_{1,k})^2\right)^{\frac{\theta}{\theta+1}}$$
$$\le \left(\sum_{k=1}^{\infty}\frac{1}{(e^2_{2,k}-e^1_{2,k}e^2_{1,k}/e^1_{1,k})^{2\theta}}\Big(\frac{g_{2,k}-g_{1,k}e^2_{1,k}/e^1_{1,k}}{e^2_{2,k}-e^1_{2,k}e^2_{1,k}/e^1_{1,k}}\Big)^2\right)^{\frac{1}{\theta+1}}$$
$$\times\left(\Big(\sum_{k=1}^{\infty}(g_{2,k})^2\Big)^{\frac{1}{2}}+\Big(\sum_{k=1}^{\infty}(e^2_{1,k}/e^1_{1,k})^2(g_{1,k})^2\Big)^{\frac{1}{2}}\right)^{\frac{2\theta}{\theta+1}}$$
$$\le \left(\Big(\frac{1}{\frac{E_{\alpha,1}(-\lambda_1 T_1^\alpha)}{E_{\alpha,1}(-\lambda_1 T_2^\alpha)}-1}\frac{\Gamma(1-\alpha)T_2^\alpha}{c}\Big)^\theta\|u^0\|_{(D(L)^\theta)}\right)^{\frac{2}{\theta+1}}\left(\|g_2\|+\Big(\frac{1}{1-E_{\alpha,1}(-\lambda_1 T_1^\alpha)}\Big)\|g_1\|\right)^{\frac{2\theta}{\theta+1}}.$$

This completes the proof. □

B. Proof of Lemma 4.1

$$
\begin{aligned}
e^1_{1,k}e^2_{2,k}-e^2_{1,k}e^1_{2,k} &= \frac{1-E_{\alpha,1}(-\lambda_k T_1^\alpha)}{\lambda_k}E_{\alpha,1}(-\lambda_k T_2^\alpha)-\frac{1-E_{\alpha,1}(-\lambda_k T_2^\alpha)}{\lambda_k}E_{\alpha,1}(-\lambda_k T_1^\alpha)\\
&= \frac{E_{\alpha,1}(-\lambda_k T_2^\alpha)-E_{\alpha,1}(-\lambda_k T_1^\alpha)}{\lambda_k},\\
e^1_{1,k}e^1_{2,k}+e^2_{1,k}e^2_{2,k} &= \frac{1-E_{\alpha,1}(-\lambda_k T_1^\alpha)}{\lambda_k}E_{\alpha,1}(-\lambda_k T_1^\alpha)+\frac{1-E_{\alpha,1}(-\lambda_k T_2^\alpha)}{\lambda_k}E_{\alpha,1}(-\lambda_k T_2^\alpha)\\
&= \frac{E_{\alpha,1}(-\lambda_k T_2^\alpha)+E_{\alpha,1}(-\lambda_k T_1^\alpha)-E^2_{\alpha,1}(-\lambda_k T_2^\alpha)-E^2_{\alpha,1}(-\lambda_k T_1^\alpha)}{\lambda_k},\\
(e^2_{1,k})^2+(e^2_{2,k})^2 &= \left(\frac{1-E_{\alpha,1}(-\lambda_k T_2^\alpha)}{\lambda_k}\right)^2+\big(E_{\alpha,1}(-\lambda_k T_2^\alpha)\big)^2,\\
(e^1_{1,k})^2+(e^2_{1,k})^2 &= \left(\frac{1-E_{\alpha,1}(-\lambda_k T_1^\alpha)}{\lambda_k}\right)^2+\left(\frac{1-E_{\alpha,1}(-\lambda_k T_2^\alpha)}{\lambda_k}\right)^2.
\end{aligned}
$$

Then we can verify that

$$
\begin{aligned}
\lim_{k\to+\infty}\lambda_k^2\omega_1(k) &= \lim_{k\to+\infty}\frac{\lambda_k(E_{\alpha,1}(-\lambda_k T_2^\alpha)-E_{\alpha,1}(-\lambda_k T_1^\alpha))^2}{(E_{\alpha,1}(-\lambda_k T_2^\alpha)+E_{\alpha,1}(-\lambda_k T_1^\alpha)-E^2_{\alpha,1}(-\lambda_k T_2^\alpha)-E^2_{\alpha,1}(-\lambda_k T_1^\alpha))}\\
&= \frac{(T_1^\alpha-T_2^\alpha)^2}{\Gamma(1-\alpha)T_1^\alpha T_2^\alpha(T_1^\alpha+T_2^\alpha)},
\end{aligned}
$$

$$
\lim_{k\to+\infty}\lambda_k^2\omega_2(k) = \lim_{k\to+\infty}\frac{(E_{\alpha,1}(-\lambda_k T_2^\alpha)-E_{\alpha,1}(-\lambda_k T_1^\alpha))^2}{\left(\frac{1-E_{\alpha,1}(-\lambda_k T_2^\alpha)}{\lambda_k}\right)^2+\big(E_{\alpha,1}(-\lambda_k T_2^\alpha)\big)^2} = \frac{T_2^{2\alpha}}{1+\Gamma^2(1-\alpha)T_2^{2\alpha}}\left(\frac{1}{T_2^\alpha}-\frac{1}{T_1^\alpha}\right)^2,
$$

and

$$
\lim_{k\to+\infty}\lambda_k^2\omega_3(k) = \lim_{k\to+\infty}\frac{(E_{\alpha,1}(-\lambda_k T_2^\alpha)-E_{\alpha,1}(-\lambda_k T_1^\alpha))^2}{\left(\frac{1-E_{\alpha,1}(-\lambda_k T_1^\alpha)}{\lambda_k}\right)^2+\left(\frac{1-E_{\alpha,1}(-\lambda_k T_2^\alpha)}{\lambda_k}\right)^2} = \frac{1}{\Gamma^2(1-\alpha)}\left(\frac{1}{T_2^\alpha}-\frac{1}{T_1^\alpha}\right)^2,
$$

which completes the proof. □

C. Proof of Theorem 4.1

Taking $\gamma=\mu$, by singular value decomposition for the compact operator as in [16] we obtain

$$
\left\{
\begin{aligned}
f_{\mu,i} &= \frac{(\mu e^1_{1,i}+(e^1_{1,i}e^2_{2,i}-e^2_{1,i}e^1_{2,i})e^2_{2,i})g_{1,i}+(\mu e^2_{1,i}+(e^2_{1,i}e^1_{2,i}-e^1_{1,i}e^2_{2,i})e^1_{2,i})g_{2,i}}{(e^1_{1,i}e^2_{2,i}-e^2_{1,i}e^1_{2,i})^2+\mu((e^1_{2,i})^2+(e^2_{2,i})^2)+\mu((e^1_{1,i})^2+(e^2_{1,i})^2)+\mu^2},\\
u^0_{\mu,i} &= \frac{(\mu e^1_{2,i}+(e^2_{1,i}e^1_{2,i}-e^2_{2,i}e^1_{1,i})e^2_{1,i})g_{1,i}+(\mu e^2_{2,i}+(e^1_{1,i}e^2_{2,i}-e^1_{2,i}e^2_{1,i})e^1_{1,i})g_{2,i}}{(e^1_{1,i}e^2_{2,i}-e^2_{1,i}e^1_{2,i})^2+\mu((e^1_{2,i})^2+(e^2_{2,i})^2)+\mu((e^1_{1,i})^2+(e^2_{1,i})^2)+\mu^2},
\end{aligned}
\right. \tag{C.1}
$$

where $g_{l,i}=(g_l(x),\varphi_i(x)),\ l=1,2,\ i=1,\cdots,+\infty$. If the observation data has noise, we have

$$\begin{cases} f^{\delta}_{\mu,i}=\dfrac{(\mu e^1_{1,i}+(e^1_{1,i}e^2_{2,i}-e^2_{1,i}e^1_{2,i})e^2_{2,i})g^{\delta}_{1,i}+(\mu e^2_{1,i}+(e^2_{1,i}e^1_{2,i}-e^1_{1,i}e^2_{2,i})e^1_{2,i})g^{\delta}_{2,i}}{(e^1_{1,i}e^2_{2,i}-e^2_{1,i}e^1_{2,i})^2+\mu((e^1_{2,i})^2+(e^2_{2,i})^2)+\mu((e^1_{1,i})^2+(e^2_{1,i})^2)+\mu^2}, \\ u^{0,\delta}_{\mu,i}=\dfrac{(\mu e^1_{2,i}+(e^2_{1,i}e^1_{2,i}-e^2_{2,i}e^1_{1,i})e^2_{1,i})g^{\delta}_{1,i}+(\mu e^2_{2,i}+(e^1_{1,i}e^2_{2,i}-e^2_{2,i}e^1_{1,i})e^1_{1,i})g^{\delta}_{2,i}}{(e^1_{1,i}e^2_{2,i}-e^2_{1,i}e^1_{2,i})^2+\mu((e^1_{2,i})^2+(e^2_{2,i})^2)+\mu((e^1_{1,i})^2+(e^2_{1,i})^2)+\mu^2}, \end{cases} \tag{C.2}$$

where $g^{\delta}_{l,i}=(g^{\delta}_l(x),\varphi_i(x)),\ l=1,2,\ i=1,\cdots,+\infty$. From the triangle inequality,

$$\left\|f^{\delta}_{\mu}(x)-f(x)\right\|\le\left\|f^{\delta}_{\mu}(x)-f_{\mu}(x)\right\|+\left\|f_{\mu}(x)-f(x)\right\| \tag{C.3}$$

and

$$\left\|u^{0,\delta}_{\mu}(x)-u^0(x)\right\|\le\left\|u^{0,\delta}_{\mu}(x)-u^0_{\mu}(x)\right\|+\left\|u^0_{\mu}(x)-u^0(x)\right\|. \tag{C.4}$$

Using Eqs. (C.1) and (C.2), we can estimate the first term of the two inequalities as follows:

$$\begin{aligned} &\left\|f^{\delta}_{\mu}(x)-f_{\mu}(x)\right\|=\left\|\sum_{k=1}^{+\infty}(f^{\delta}_{\mu,k}-f_{\mu,k})\varphi_k(x)\right\|\le\frac{\delta}{\sqrt{\mu}}, \\ &\left\|u^{0,\delta}_{\mu}(x)-u^0_{\mu}(x)\right\|=\left\|\sum_{k=1}^{+\infty}(u^{0,\delta}_{\mu,k}-u^0_{\mu,k})\varphi_k(x)\right\|\le\frac{\delta}{\sqrt{\mu}}. \end{aligned} \tag{C.5}$$

Next, we estimate the second term of the inequality (C.3). Subtracting (C.1), from (A.1) we obtain

$$\begin{aligned} &\left\|f_{\mu}(x)-f(x)\right\| \\ =&\left\|\sum_{k=1}^{+\infty}\frac{\mu^2 f_k+\mu((e^1_{2,k})^2+(e^2_{2,k})^2)f_k-\mu(e^1_{1,k}e^1_{2,k}+e^2_{1,k}e^2_{2,k})u^0_k}{\mu^2+\mu((e^1_{1,k})^2+(e^2_{1,k})^2)+\mu((e^1_{2,k})^2+(e^2_{2,k})^2)+(e^1_{1,k}e^2_{2,k}-e^2_{1,k}e^1_{2,k})^2}\varphi_k(x)\right\| \\ \le&\left\|\sum_{k=1}^{+\infty}\frac{\mu f_k}{\mu+\frac{\mu((e^1_{1,k})^2+(e^2_{1,k})^2)+(e^1_{1,k}e^2_{2,k}-e^2_{1,k}e^1_{2,k})^2}{\mu+(e^1_{2,k})^2+(e^2_{2,k})^2}}\right\|+\left\|\sum_{k=1}^{+\infty}\frac{\mu u^0_k}{2\mu+\frac{(e^1_{1,k}e^2_{2,k}-e^2_{1,k}e^1_{2,k})^2}{e^1_{1,k}e^1_{2,k}+e^2_{1,k}e^2_{2,k}}}\varphi_k(x)\right\| \\ \le&\left\|\sum_{k=1}^{+\infty}\frac{\mu f_k}{\mu+\frac{(e^1_{1,k}e^2_{2,k}-e^2_{1,k}e^1_{2,k})^2}{(e^1_{2,k})^2+(e^2_{2,k})^2}}\right\|+\sum_{k=1}^{+\infty}\left\|\frac{\mu u^0_k}{\mu+\frac{(e^1_{1,k}e^2_{2,k}-e^2_{1,k}e^1_{2,k})^2}{e^1_{1,k}e^1_{2,k}+e^2_{1,k}e^2_{2,k}}}\varphi_k(x)\right\|. \end{aligned}$$

From Lemma 4.1, there exist some positive constants $C_1,\ C_2$ satisfying $\frac{(e^1_{1,k}e^2_{2,k}-e^2_{1,k}e^1_{2,k})^2}{(e^1_{2,k})^2+(e^2_{2,k})^2}\geq\frac{C_1}{\lambda_k^2}$ and $\frac{(e^1_{1,k}e^2_{2,k}-e^2_{1,k}e^1_{2,k})^2}{e^1_{1,k}e^1_{2,k}+e^2_{1,k}e^2_{2,k}}\geq\frac{C_2}{\lambda_k^2}$. Therefore

$$\begin{aligned}\left\|f_\mu(x)-f(x)\right\| &\leq \left\|\sum_{k=1}^{+\infty}\frac{\mu f_k}{\mu+\frac{C_1}{\lambda_k^2}}\varphi_k(x)\right\|+\left\|\sum_{k=1}^{+\infty}\frac{\mu u_k^0}{\mu+\frac{C_2}{\lambda_k^2}}\varphi_k(x)\right\|\\ &\leq \sup_{k\in\mathbb{N}^+}(A(k))\left\|\sum_{k=1}^{+\infty}f_k\lambda_k^\theta\varphi_k(x)\right\|+\sup_{k\in\mathbb{N}^+}(B(k))\left\|\sum_{k=1}^{+\infty}\lambda_k^\theta u_k^0\varphi_k(x)\right\|,\end{aligned}$$

where

$$A(k)=\frac{\mu\lambda_k^{2-\theta}}{\mu\lambda_k^2+C_1},\qquad B(k)=\frac{\mu\lambda_k^{2-\theta}}{\mu\lambda_k^2+C_2}.$$

Similarly,

$$\begin{aligned}&\left\|u_\mu^0(x)-u^0(x)\right\|\\ =&\left\|\sum_{k=1}^{+\infty}\frac{\mu^2u_k^0+\mu((e^1_{1,k})^2+(e^2_{1,k})^2)u_k^0-\mu(e^1_{1,k}e^1_{2,k}+e^2_{1,k}e^2_{2,k})f_k}{\mu^2+\mu((e^1_{1,k})^2+(e^2_{1,k})^2)+\mu((e^1_{2,k})^2+(e^2_{2,k})^2)+(e^1_{1,k}e^2_{2,k}-e^2_{1,k}e^1_{2,k})^2}\varphi_k(x)\right\|\\ \leq&\left\|\sum_{k=1}^{+\infty}\frac{\mu u_k^0}{\mu+\frac{\mu((e^1_{2,k})^2+(e^2_{2,k})^2)+(e^1_{1,k}e^2_{2,k}-e^2_{1,k}e^1_{2,k})^2}{\mu+(e^1_{1,k})^2+(e^2_{1,k})^2}}\varphi_k(x)\right\|+\left\|\sum_{k=1}^{+\infty}\frac{\mu f_k}{\mu+\frac{(e^1_{1,k}e^2_{2,k}-e^2_{1,k}e^1_{2,k})^2}{e^1_{1,k}e^1_{2,k}+e^2_{1,k}e^2_{2,k}}}\varphi_k(x)\right\|\\ \leq&\left\|\sum_{k=1}^{+\infty}\frac{\mu u_k^0}{\mu+\frac{(e^1_{1,k}e^2_{2,k}-e^2_{1,k}e^1_{2,k})^2}{(e^1_{1,k})^2+(e^2_{1,k})^2}}\varphi_k(x)\right\|+\left\|\sum_{k=1}^{+\infty}\frac{\mu f_k}{\mu+\frac{(e^1_{1,k}e^2_{2,k}-e^2_{1,k}e^1_{2,k})^2}{e^1_{1,k}e^1_{2,k}+e^2_{1,k}e^2_{2,k}}}\varphi_k(x)\right\|\\ \leq&\left\|\sum_{k=1}^{+\infty}\frac{\mu f_k}{\mu+\frac{C_2}{\lambda_k^2}}\varphi_k(x)\right\|+\left\|\sum_{k=1}^{+\infty}\frac{\mu u_k^0}{\mu+\frac{C_3}{\lambda_k^2}}\varphi_k(x)\right\|\\ \leq&\sup_{k\in\mathbb{N}^+}(B(k))\left\|\sum_{k=1}^{+\infty}f_k\lambda_k^\theta\varphi_k(x)\right\|+\sup_{k\in\mathbb{N}^+}(C(k))\left\|\sum_{k=1}^{+\infty}\lambda_k^\theta u_k^0\varphi_k(x)\right\|,\end{aligned}$$

where

$$C(k)=\frac{\mu\lambda_k^{2-\theta}}{\mu\lambda_k^2+C_3}.$$

For constant $p>0$, $\mu>0$, $\beta>0$ and $s\geq\lambda_1>0$, we claim that

$$F(s)=\frac{\mu s^{2-p}}{\beta+\mu s^2}\leq\begin{cases}C\mu^{\frac{p}{2}}, & 0<p<2,\\ C\mu, & p\geq 2,\end{cases}$$

whence

$$\left\|f_\mu(x)-f(x)\right\| = \begin{cases} C_4E\mu^{\frac{\theta}{2}}, & 0<\theta<2\,, \\ C_5E\mu\,, & \theta\geq 2\,, \end{cases}$$

$$\left\|u^0_\mu(x)-u^0(x)\right\| = \begin{cases} C_6E\mu^{\frac{\theta}{2}}\,, & 0<\theta<2\,, \\ C_7E\mu\,, & \theta\geq 2\,. \end{cases}$$

From inequalities (C.3) and (C.4),

$$\left\|f^\delta_\mu(x)-f(x)\right\| \leq \frac{\delta}{\sqrt{\mu}} + \begin{cases} C_4E\mu^{\frac{\theta}{2}}, & 0<\theta<2\,, \\ C_5E\mu, & \theta\geq 2\,, \end{cases}$$

$$\left\|u^{0,\delta}_\mu(x)-u^0(x)\right\| \leq \frac{\delta}{\sqrt{\mu}} + \begin{cases} C_6E\mu^{\frac{\theta}{2}}, & 0<\theta<2\,, \\ C_7E\mu, & \theta\geq 2\,, \end{cases}$$

where $\{C_i\,, i=4,5,6,7\}$ is relative to $\{\theta,\lambda_1,T_1,T_2,\alpha\}$. If we choose

$$\mu = \begin{cases} \left(\dfrac{\delta}{E}\right)^{\frac{2}{\theta+1}}\,, & 0<\theta<2\,, \\ \left(\dfrac{\delta}{E}\right)^{\frac{2}{3}}\,, & \theta\geq 2\,, \end{cases}$$

then Theorem 4.1 holds. □

D. Proof of Lemma 4.4

By definition,

$$\sum_{k=1}^{\infty}\frac{1-E_{\alpha,1}(-\lambda_kT_i^\alpha)}{\lambda_k}f_k\varphi_k(x)+\sum_{k=1}^{\infty}E_{\alpha,1}(-\lambda_kT_i^\alpha)u_k^0\varphi_k(x)=\sum_{k=1}^{\infty}g_{i,k}\varphi_k(x)\,,\ i=1,2\,. \quad \text{(D.1)}$$

Denoting $\Delta f_k = f_k^\delta - f_k$, $\Delta u_k^0 = u_k^{0,\delta} - u_k^0$, $\Delta g_{1,k} = g_{1,k}^\delta - g_{1,k}$, $\Delta g_{2,k} = g_{2,k}^\delta - g_{2,k}$ for $k=1,\cdots,\infty$, we obtain

$$\sum_{k=1}^{\infty}\frac{1-E_{\alpha,1}(-\lambda_kT_1^\alpha)}{\lambda_k}\Delta f_k\varphi_k(x)+\sum_{k=1}^{\infty}E_{\alpha,1}(-\lambda_kT_1^\alpha)\Delta u_k^0\varphi_k(x)=\sum_{k=1}^{\infty}\Delta g_{1,k}\varphi_k(x)\,,$$

$$\sum_{k=1}^{\infty}\frac{1-E_{\alpha,1}(-\lambda_kT_2^\alpha)}{\lambda_k}\Delta f_k\varphi_k(x)+\sum_{k=1}^{\infty}E_{\alpha,1}(-\lambda_kT_2^\alpha)\Delta u_k^0\varphi_k(x)=\sum_{k=1}^{\infty}\Delta g_{2,k}\varphi_k(x)\,.$$

From simple computation,

$$\Delta f_k = \frac{\lambda_k\left(\Delta g_{1,k}E_{\alpha,1}(-\lambda_kT_2^\alpha)-\Delta g_{2,k}E_{\alpha,1}(-\lambda_kT_1^\alpha)\right)}{E_{\alpha,1}(-\lambda_kT_2^\alpha)-E_{\alpha,1}(-\lambda_kT_1^\alpha)}\,,$$

$$\Delta u_k^0 = \frac{\Delta g_{1,k} E_{\alpha,1}(-\lambda_k T_2^\alpha) - \Delta g_{2,k} E_{\alpha,1}(-\lambda_k T_1^\alpha) + \Delta g_{2,k} - \Delta g_{1,k}}{E_{\alpha,1}(-\lambda_k T_2^\alpha) - E_{\alpha,1}(-\lambda_k T_1^\alpha)} .$$

Writing

$$\Delta f(x) = \sum_{k=1}^{\infty} \Delta f_k \varphi_k(x) = \sum_{k=1}^{\infty} \frac{\lambda_k \left(\Delta g_{1,k} E_{\alpha,1}(-\lambda_k T_2^\alpha) - \Delta g_{2,k} E_{\alpha,1}(-\lambda_k T_1^\alpha)\right)}{E_{\alpha,1}(-\lambda_k T_2^\alpha) - E_{\alpha,1}(-\lambda_k T_1^\alpha)} \varphi_k(x) ,$$

$$\Delta u^0(x) = \sum_{k=1}^{\infty} \Delta u_k^0 \varphi_k(x) = \sum_{k=1}^{\infty} \frac{\Delta g_{1,k} E_{\alpha,1}(-\lambda_k T_2^\alpha) - \Delta g_{2,k} E_{\alpha,1}(-\lambda_k T_1^\alpha) + \Delta g_{2,k} - \Delta g_{1,k}}{E_{\alpha,1}(-\lambda_k T_2^\alpha) - E_{\alpha,1}(-\lambda_k T_1^\alpha)} \varphi_k(x) ,$$

we obtain

$$\begin{aligned} &\left(K_{1,1}\Delta f\right)(x) = \sum_{k=1}^{\infty} \Delta f_k K_{1,1} \varphi_k(x) \\ =&\sum_{k=1}^{\infty} \frac{\lambda_k \left(\Delta g_{1,k} E_{\alpha,1}(-\lambda_k T_2^\alpha) - \Delta g_{2,k} E_{\alpha,1}(-\lambda_k T_1^\alpha)\right)}{E_{\alpha,1}(-\lambda_k T_2^\alpha) - E_{\alpha,1}(-\lambda_k T_1^\alpha)} e_{1,k}^1 \varphi_k(x) , \end{aligned}$$

$$\begin{aligned} &\left\|(K_{1,1}\Delta f)(x)\right\|_{L^2(\Omega)} \\ =&\left\| \sum_{k=1}^{\infty} \frac{(1 - E_{\alpha,1}(-\lambda_k T_1^\alpha))(\Delta g_{1,k} E_{\alpha,1}(-\lambda_k T_2^\alpha) - \Delta g_{2,k} E_{\alpha,1}(-\lambda_k T_1^\alpha))}{E_{\alpha,1}(-\lambda_k T_2^\alpha) - E_{\alpha,1}(-\lambda_k T_1^\alpha)} \varphi_k(x) \right\|_{L^2(\Omega)} \\ \le& \left\| \sum_{k=1}^{\infty} \frac{(1 - E_{\alpha,1}(-\lambda_k T_1^\alpha))\Delta g_{1,k} E_{\alpha,1}(-\lambda_k T_2^\alpha)}{E_{\alpha,1}(-\lambda_k T_2^\alpha) - E_{\alpha,1}(-\lambda_k T_1^\alpha)} \varphi_k(x) \right\|_{L^2(\Omega)} \\ &+ \left\| \sum_{k=1}^{\infty} \frac{(1 - E_{\alpha,1}(-\lambda_k T_1^\alpha))\Delta g_{2,k} E_{\alpha,1}(-\lambda_k T_1^\alpha)}{E_{\alpha,1}(-\lambda_k T_2^\alpha) - E_{\alpha,1}(-\lambda_k T_1^\alpha)} \varphi_k(x) \right\|_{L^2(\Omega)} . \end{aligned}$$

From Lemmas 2.1 and 4.3,

$$\left| \frac{(1 - E_{\alpha,1}(-\lambda_k T_1^\alpha)) E_{\alpha,1}(-\lambda_k T_2^\alpha)}{E_{\alpha,1}(-\lambda_k T_2^\alpha) - E_{\alpha,1}(-\lambda_k T_1^\alpha)} \right| \le \frac{E_{\alpha,1}(-\lambda_1 T_2^\alpha)}{E_{\alpha,1}(-\lambda_1 T_1^\alpha) - E_{\alpha,1}(-\lambda_1 T_2^\alpha)} ,$$

hence

$$\left\|(K_{1,1}\Delta f)(x)\right\|_{L^2(\Omega)} \le \frac{E_{\alpha,1}(-\lambda_1 T_2^\alpha) + E_{\alpha,1}(-\lambda_1 T_1^\alpha)}{E_{\alpha,1}(-\lambda_1 T_1^\alpha) - E_{\alpha,1}(-\lambda_1 T_2^\alpha)} \delta .$$

Now

$$\begin{aligned} &\left(K_{1,2}\Delta f\right)(x) = \sum_{k=1}^{\infty} \Delta f_k K_{1,2} \varphi_k(x) \\ =&\sum_{k=1}^{\infty} \frac{\lambda_k (\Delta g_{1,k} E_{\alpha,1}(-\lambda_k T_2^\alpha) - \Delta g_{2,k} E_{\alpha,1}(-\lambda_k T_1^\alpha))}{E_{\alpha,1}(-\lambda_k T_2^\alpha) - E_{\alpha,1}(-\lambda_k T_1^\alpha)} e_{1,k}^2 \varphi_k(x), \end{aligned}$$

$$
\begin{aligned}
&\left\|(K_{1,2}\Delta f)(x)\right\|_{L^2(\Omega)} \\
=&\left\|\sum_{k=1}^{\infty}\frac{(1-E_{\alpha,1}(-\lambda_k T_2^\alpha))(\Delta g_{1,k}E_{\alpha,1}(-\lambda_k T_2^\alpha)-\Delta g_{2,k}E_{\alpha,1}(-\lambda_k T_1^\alpha))}{E_{\alpha,1}(-\lambda_k T_2^\alpha)-E_{\alpha,1}(-\lambda_k T_1^\alpha)}\varphi_k(x)\right\|_{L^2(\Omega)} \\
\le&\left\|\sum_{k=1}^{\infty}\frac{(1-E_{\alpha,1}(-\lambda_k T_2^\alpha))\Delta g_{1,k}E_{\alpha,1}(-\lambda_k T_2^\alpha)}{E_{\alpha,1}(-\lambda_k T_2^\alpha)-E_{\alpha,1}(-\lambda_k T_1^\alpha)}\varphi_k(x)\right\|_{L^2(\Omega)} \\
&+\left\|\sum_{k=1}^{\infty}\frac{(1-E_{\alpha,1}(-\lambda_k T_2^\alpha))\Delta g_{2,k}E_{\alpha,1}(-\lambda_k T_1^\alpha)}{E_{\alpha,1}(-\lambda_k T_2^\alpha)-E_{\alpha,1}(-\lambda_k T_1^\alpha)}\varphi_k(x)\right\|_{L^2(\Omega)},
\end{aligned}
$$

so from Lemmas 2.1 and 4.3 we have

$$
\left|\frac{(1-E_{\alpha,1}(-\lambda_k T_2^\alpha))E_{\alpha,1}(-\lambda_k T_2^\alpha)}{E_{\alpha,1}(-\lambda_k T_2^\alpha)-E_{\alpha,1}(-\lambda_k T_1^\alpha)}\right| \le \frac{E_{\alpha,1}(-\lambda_1 T_2^\alpha)}{E_{\alpha,1}(-\lambda_1 T_1^\alpha)-E_{\alpha,1}(-\lambda_1 T_2^\alpha)},
$$

and hence

$$
\left\|(K_{1,2}\Delta f)(x)\right\|_{L^2(\Omega)} \le \frac{E_{\alpha,1}(-\lambda_1 T_2^\alpha)+E_{\alpha,1}(-\lambda_1 T_1^\alpha)}{E_{\alpha,1}(-\lambda_1 T_1^\alpha)-E_{\alpha,1}(-\lambda_1 T_2^\alpha)}\delta .
$$

By a similar argument, we can prove that

$$
\begin{aligned}
&\left\|K_{2,1}(\Delta u^0)\right\| \le \frac{3E_{\alpha,1}(-\lambda_1 T_1^\alpha)+E_{\alpha,1}(-\lambda_1 T_2^\alpha)}{E_{\alpha,1}(-\lambda_1 T_1^\alpha)-E_{\alpha,1}(-\lambda_1 T_2^\alpha)}\delta, \\
&\left\|K_{2,2}(\Delta u^0)\right\| \le \frac{4E_{\alpha,1}(-\lambda_1 T_1^\alpha)}{E_{\alpha,1}(-\lambda_1 T_1^\alpha)-E_{\alpha,1}(-\lambda_1 T_2^\alpha)}\delta, \\
&\left\|K_{1,j}(f_\mu^\delta-f^\delta)\right\| \le C\delta, \qquad \left\|K_{2,j}(u_\mu^{0,\delta}-u^{0,\delta})\right\| \le C\delta, \qquad j=1,2,
\end{aligned}
$$

which complete the proof. □

E. Proof of Lemma 4.5

Subtracting Eqs. (C.1) from Eqs. (A.1),

$$
\begin{aligned}
f_{\mu,i}-f_i &= \frac{-\mu(((e^1_{2,i})^2+(e^2_{2,i})^2+(e^2_{1,i})^2+\mu)e^2_{2,i}+e^1_{1,i}e^2_{1,i}e^1_{2,i})g_{1,i}+\mu(((e^1_{2,i})^2+(e^2_{2,i})^2+(e^1_{1,i})^2+\mu)e^1_{2,i}+e^1_{1,i}e^2_{1,i}e^2_{2,i})g_{2,i}}{((e^1_{1,i}e^2_{2,i}-e^2_{1,i}e^1_{2,i})^2+\mu((e^1_{2,i})^2+(e^2_{2,i})^2+(e^1_{1,i})^2+(e^2_{1,i})^2+\mu))(e^1_{1,i}e^2_{2,i}-e^2_{1,i}e^1_{2,i})}, \\
f^\delta_{\mu,i}-f^\delta_i &= \frac{-\mu(((e^1_{2,i})^2+(e^2_{2,i})^2+(e^2_{1,i})^2+\mu)e^2_{2,i}+e^1_{1,i}e^2_{1,i}e^1_{2,i})g^\delta_{1,i}+\mu(((e^1_{2,i})^2+(e^2_{2,i})^2+(e^1_{1,i})^2+\mu)e^1_{2,i}+e^1_{1,i}e^2_{1,i}e^2_{2,i})g^\delta_{2,i}}{((e^1_{1,i}e^2_{2,i}-e^2_{1,i}e^1_{2,i})^2+\mu((e^1_{2,i})^2+(e^2_{2,i})^2+(e^1_{1,i})^2+(e^2_{1,i})^2+\mu))(e^1_{1,i}e^2_{2,i}-e^2_{1,i}e^1_{2,i})},
\end{aligned}
$$

hence

$$
\begin{aligned}
(f^\delta_{\mu,i}-f^\delta_i)-(f_{\mu,i}-f_i) =& \frac{-\mu(((e^1_{2,i})^2+(e^2_{2,i})^2+(e^2_{1,i})^2+\mu)e^2_{2,i}+e^1_{1,i}e^2_{1,i}e^1_{2,i})(g^\delta_{1,i}-g_{1,i})}{((e^1_{1,i}e^2_{2,i}-e^2_{1,i}e^1_{2,i})^2+\mu((e^1_{2,i})^2+(e^2_{2,i})^2+(e^1_{1,i})^2+(e^2_{1,i})^2+\mu))(e^1_{1,i}e^2_{2,i}-e^2_{1,i}e^1_{2,i})} \\
&+\frac{\mu(((e^1_{2,i})^2+(e^2_{2,i})^2+(e^1_{1,i})^2+\mu)e^1_{2,i}+e^1_{1,i}e^2_{1,i}e^2_{2,i})(g^\delta_{2,i}-g_{2,i})}{((e^1_{1,i}e^2_{2,i}-e^2_{1,i}e^1_{2,i})^2+\mu((e^1_{2,i})^2+(e^2_{2,i})^2+(e^1_{1,i})^2+(e^2_{1,i})^2+\mu))(e^1_{1,i}e^2_{2,i}-e^2_{1,i}e^1_{2,i})},
\end{aligned}
$$

so

$$
\begin{aligned}
&\Big\|K_{1,1}((f_\mu^\delta - f^\delta)-(f_\mu - f))\Big\| \\
=&\Bigg\|\sum_{i=1}^{\infty}\Bigg(\frac{-\mu(((e_{2,i}^1)^2+(e_{2,i}^2)^2+(e_{1,i}^2)^2+\mu)e_{2,i}^2+e_{1,i}^1e_{1,i}^2e_{2,i}^1)(g_{1,i}^\delta-g_{1,i})}{((e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1)^2+\mu((e_{2,i}^1)^2+(e_{2,i}^2)^2+(e_{1,i}^1)^2+(e_{1,i}^2)^2+\mu))(e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1)}\Bigg)e_{1,i}^1\varphi_i(x) \\
&+\Bigg(\frac{\mu(((e_{2,i}^1)^2+(e_{2,i}^2)^2+(e_{1,i}^1)^2+\mu)e_{2,i}^1+e_{1,i}^1e_{1,i}^2e_{2,i}^2)(g_{2,i}^\delta-g_{2,i})}{((e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1)^2+\mu((e_{2,i}^1)^2+(e_{2,i}^2)^2+(e_{1,i}^1)^2+(e_{1,i}^2)^2+\mu))(e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1)}\Bigg)e_{1,i}^1\varphi_i(x)\Bigg\| \\
\le&\Bigg\|\sum_{i=1}^{\infty}\frac{e_{2,i}^2e_{1,i}^1(g_{1,i}^\delta-g_{1,i})}{e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1}\Bigg\|+\Bigg\|\sum_{i=1}^{\infty}\frac{e_{1,i}^2e_{2,i}^1(g_{1,i}^\delta-g_{1,i})}{e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1}\Bigg\|+\Bigg\|\sum_{i=1}^{\infty}\frac{e_{2,i}^1e_{1,i}^1(g_{2,i}^\delta-g_{2,i})}{e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1}\Bigg\|+\Bigg\|\sum_{i=1}^{\infty}\frac{e_{1,i}^2e_{2,i}^2(g_{2,i}^\delta-g_{2,i})}{e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1}\Bigg\|\,.
\end{aligned}
$$

It is easy to prove the limits of $\frac{e_{2,i}^2e_{1,i}^1}{e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1}$, $\frac{e_{1,i}^2e_{2,i}^1}{e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1}$, $\frac{e_{2,i}^1e_{1,i}^1}{e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1}$, $\frac{e_{1,i}^2e_{2,i}^2}{e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1}$ exist, so there exists some nonnegative constant C_8 such that $\|K_{1,1}((f_\mu^\delta-f^\delta)-(f_\mu-f))\|\le C_8\delta$. As $\|K_{1,1}(f_\mu-f)\|\le\|K_{1,1}(f_\mu^\delta-f^\delta)\|+\|K_{1,1}((f_\mu^\delta-f^\delta)-(f_\mu-f))\|$, we get $\|K_{1,1}(f_\mu-f)\|\le C_9\delta$.

Similarly we can prove that

$$
\begin{aligned}
&\Big\|K_{2,1}(u_\mu^0-u^0)\Big\|\le C_{10}\delta\,,\quad \Big\|K_{1,2}(f_\mu-f)\Big\|\le C_{11}\delta\,,\quad \Big\|K_{2,2}(u_\mu^0-u^0)\Big\|\le C_{12}\delta\,,\\
&\Big\|f_\mu-f\Big\|_{(D(L)^\theta)}=\Bigg(\sum_{i=1}^{\infty}\lambda_i^{2\theta}\Bigg(\frac{-\mu(((e_{2,i}^1)^2+(e_{2,i}^2)^2+(e_{1,i}^2)^2+\mu)e_{2,i}^2+e_{1,i}^1e_{1,i}^2e_{2,i}^1)g_{1,i}}{((e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1)^2+\mu((e_{2,i}^1)^2+(e_{2,i}^2)^2+(e_{1,i}^1)^2+(e_{1,i}^2)^2+\mu))(e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1)}\\
&\qquad+\frac{\mu(((e_{2,i}^1)^2+(e_{2,i}^2)^2+(e_{1,i}^1)^2+\mu)e_{2,i}^1+e_{1,i}^1e_{1,i}^2e_{2,i}^2)g_{2,i}}{((e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1)^2+\mu((e_{2,i}^1)^2+(e_{2,i}^2)^2+(e_{1,i}^1)^2+(e_{1,i}^2)^2+\mu))(e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1)}\Bigg)^2\Bigg)^{\frac{1}{2}}\\
&\qquad\le\Bigg(\sum_{i=1}^{\infty}\lambda_i^{2\theta}\Bigg(\frac{-\mu(((e_{2,i}^1)^2+(e_{2,i}^2)^2+(e_{1,i}^2)^2+\mu)e_{2,i}^2+e_{1,i}^1e_{1,i}^2e_{2,i}^1)g_{1,i}}{((e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1)^2+\mu((e_{2,i}^1)^2+(e_{2,i}^2)^2+(e_{1,i}^1)^2+(e_{1,i}^2)^2+\mu))(e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1)}\Bigg)^2\Bigg)^{\frac{1}{2}}\\
&\qquad+\Bigg(\sum_{i=1}^{\infty}\lambda_i^{2\theta}\Bigg(\frac{\mu(((e_{2,i}^1)^2+(e_{2,i}^2)^2+(e_{1,i}^1)^2+\mu)e_{2,i}^1+e_{1,i}^1e_{1,i}^2e_{2,i}^2)g_{2,i}}{((e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1)^2+\mu((e_{2,i}^1)^2+(e_{2,i}^2)^2+(e_{1,i}^1)^2+(e_{1,i}^2)^2+\mu))(e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1)}\Bigg)^2\Bigg)^{\frac{1}{2}}\\
&\qquad\le\Bigg(\sum_{i=1}^{\infty}\lambda_i^{2\theta}(\frac{e_{2,i}^2g_{1,i}}{(e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1)})^2\Bigg)^{\frac{1}{2}}+\Bigg(\sum_{i=1}^{\infty}\lambda_i^{2\theta}(\frac{e_{2,i}^1g_{1,i}}{(e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1)})^2\Bigg)^{\frac{1}{2}}\\
&\qquad+\Bigg(\sum_{i=1}^{\infty}\lambda_i^{2\theta}(\frac{e_{2,i}^1g_{2,i}}{(e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1)})^2\Bigg)^{\frac{1}{2}}+\Bigg(\sum_{i=1}^{\infty}\lambda_i^{2\theta}(\frac{e_{2,i}^2g_{2,i}}{(e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1)})^2\Bigg)^{\frac{1}{2}}\,,\\
&\Big\|u_\mu^0-u^0\Big\|_{(D(L))^\theta}=\Bigg(\sum_{i=1}^{\infty}\lambda_i^{2\theta}\Bigg(\frac{\mu(((e_{1,i}^1)^2+(e_{1,i}^2)^2+(e_{2,i}^2)^2+\mu)e_{1,i}^2+e_{1,i}^1e_{2,i}^1e_{2,i}^2)g_{1,i}}{((e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1)^2+\mu((e_{2,i}^1)^2+(e_{2,i}^2)^2+(e_{1,i}^1)^2+(e_{1,i}^2)^2+\mu))(e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1)}\\
&\qquad+\frac{-\mu(((e_{1,i}^1)^2+(e_{1,i}^2)^2+(e_{2,i}^1)^2+\mu)e_{1,i}^1+e_{1,i}^2e_{2,i}^1e_{2,i}^2)g_{2,i}}{((e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1)^2+\mu((e_{2,i}^1)^2+(e_{2,i}^2)^2+(e_{1,i}^1)^2+(e_{1,i}^2)^2+\mu))(e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1)}\Bigg)^2\Bigg)^{\frac{1}{2}}\\
&\qquad\le\Bigg(\sum_{i=1}^{\infty}\lambda_i^{2\theta}(\frac{\mu(((e_{1,i}^1)^2+(e_{1,i}^2)^2+(e_{2,i}^2)^2+\mu)e_{1,i}^2+e_{1,i}^1e_{2,i}^1e_{2,i}^2)g_{1,i}}{((e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1)^2+\mu((e_{2,i}^1)^2+(e_{2,i}^2)^2+(e_{1,i}^1)^2+(e_{1,i}^2)^2+\mu))(e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1)})^2\Bigg)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\sum_{i=1}^{\infty}\lambda_i^{2\theta}\Big(\frac{-\mu(((e_{1,i}^1)^2+(e_{1,i}^2)^2+(e_{2,i}^1)^2+\mu)e_{1,i}^1+e_{1,i}^2e_{2,i}^1e_{2,i}^2)g_{2,i}}{((e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1)^2+\mu((e_{2,i}^1)^2+(e_{2,i}^2)^2+(e_{1,i}^1)^2+(e_{1,i}^2)^2+\mu))(e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1)}\Big)^2\right)^{\frac{1}{2}}\\
&\le\left(\sum_{i=1}^{\infty}\lambda_i^{2\theta}\Big(\frac{e_{1,i}^2g_{1,i}}{(e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1)}\Big)^2\right)^{\frac{1}{2}}+\left(\sum_{i=1}^{\infty}\lambda_i^{2\theta}\Big(\frac{e_{1,i}^1g_{1,i}}{(e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1)}\Big)^2\right)^{\frac{1}{2}}\\
&+\left(\sum_{i=1}^{\infty}\lambda_i^{2\theta}\Big(\frac{e_{1,i}^1g_{2,i}}{(e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1)}\Big)^2\right)^{\frac{1}{2}}+\left(\sum_{i=1}^{\infty}\lambda_i^{2\theta}\Big(\frac{e_{1,i}^2g_{2,i}}{(e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1)}\Big)^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

We can now easily obtain the limiting relationships

$$
\lim_{k\to+\infty}\left|\frac{\frac{e_{1,k}^1}{e_{1,k}^1e_{2,k}^2-e_{1,k}^2e_{2,k}^1}}{\lambda_k}\right|=\frac{\Gamma(1-\alpha)(T_1T_2)^\alpha}{T_2^\alpha-T_1^\alpha},\qquad \lim_{k\to+\infty}\left|\frac{\frac{e_{1,k}^2}{e_{1,k}^1e_{2,k}^2-e_{1,k}^2e_{2,k}^1}}{\lambda_k}\right|=\frac{\Gamma(1-\alpha)(T_1T_2)^\alpha}{T_2^\alpha-T_1^\alpha},
$$

$$
\lim_{k\to+\infty}\left|\frac{\frac{e_{2,k}^1}{e_{1,k}^1e_{2,k}^2-e_{1,k}^2e_{2,k}^1}}{\lambda_k}\right|=\frac{T_2^\alpha}{T_2^\alpha-T_1^\alpha},\qquad \lim_{k\to+\infty}\left|\frac{\frac{e_{2,k}^2}{e_{1,k}^1e_{2,k}^2-e_{1,k}^2e_{2,k}^1}}{\lambda_k}\right|=\frac{T_1^\alpha}{T_2^\alpha-T_1^\alpha},
$$

and there are some nonnegative constants C_{13} andC_{14} such that

$$
\begin{aligned}
\left\|f_\mu-f\right\|_{(D(L))^\theta}&\le\frac{C_{13}}{2}\left(\|g_1\|_{(D(L))^{\theta+1}}+\|g_2\|_{(D(L))^{\theta+1}}\right)\le C_{13}E\,,\\
\left\|u_\mu^0-u^0\right\|_{(D(L))^\theta}&\le\frac{C_{14}}{2}\left(\|g_1\|_{(D(L))^{\theta+1}}+\|g_2\|_{(D(L))^{\theta+1}}\right)\le C_{14}E\,.
\end{aligned}
$$

From Theorem 3.2, there exists some nonnegative constant C such that $\|f_\mu-f\|\le C\delta^{\frac{\theta}{\theta+1}}$, $\|u_\mu^0-u^0\|\le C\delta^{\frac{\theta}{\theta+1}}$, so Lemma 4.5 holds. □

F. Proof of Theorem 4.2

Firstly, one can find

$$
\begin{aligned}
&\left\|K_{1,1}(f_\mu^\delta)+K_{2,1}(u_\mu^{0,\delta})-g_1^\delta\|+\|K_{1,2}(f_\mu^\delta)+K_{2,2}(u_\mu^{0,\delta})-g_2^\delta\right\|\\
=&\left\|\sum_{k=1}^{\infty}\frac{-\mu((e_{1,k}^2)^2+(e_{2,k}^2)^2+\mu)g_{1,k}^\delta+\mu(e_{1,k}^2e_{1,k}^1+e_{2,k}^1e_{2,k}^2)g_{2,k}^\delta}{((e_{1,k}^1e_{2,k}^2-e_{1,k}^2e_{2,k}^1)^2+\mu((e_{1,k}^1)^2+(e_{1,k}^2)^2)+\mu((e_{2,k}^1)^2+(e_{2,k}^2)^2+\mu))}\varphi_k(x)\right\|\\
&+\left\|\sum_{k=1}^{\infty}\frac{\mu(e_{1,i}^1e_{1,i}^2+e_{2,i}^1e_{2,i}^2)g_{1,i}^\delta-\mu((e_{1,i}^1)^2+(e_{2,i}^1)^2+\mu)g_{2,i}^\delta}{(e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1)^2+\mu((e_{2,i}^1)^2+(e_{2,i}^2)^2+(e_{1,i}^1)^2+(e_{1,i}^2)^2)+\mu^2}\varphi_k(x)\right\|\\
\le&\left\|\sum_{k=1}^{\infty}\frac{-\mu((e_{1,k}^2)^2+(e_{2,k}^2)^2+\mu)(g_{1,k}^\delta-g_{1,k})}{((e_{1,k}^1e_{2,k}^2-e_{1,k}^2e_{2,k}^1)^2+\mu((e_{1,k}^1)^2+(e_{1,k}^2)^2)+\mu((e_{2,k}^1)^2+(e_{2,k}^2)^2+\mu))}\varphi_k(x)\right\|\\
&+\left\|\sum_{k=1}^{\infty}\frac{\mu(e_{1,k}^2e_{1,k}^1+e_{2,k}^1e_{2,k}^2)(g_{2,k}^\delta-g_{2,k})}{((e_{1,k}^1e_{2,k}^2-e_{1,k}^2e_{2,k}^1)^2+\mu((e_{1,k}^1)^2+(e_{1,k}^2)^2)+\mu((e_{2,k}^1)^2+(e_{2,k}^2)^2+\mu))}\varphi_k(x)\right\|
\end{aligned}
$$

$$
\begin{aligned}
&+\left\|\sum_{k=1}^{\infty}\frac{-\mu((e_{1,k}^2)^2+(e_{2,k}^2)^2+\mu)g_{1,k}+\mu(e_{1,k}^2e_{1,k}^1+e_{2,k}^1e_{2,k}^2)g_{2,k}}{((e_{1,k}^1e_{2,k}^2-e_{1,k}^2e_{2,k}^1)^2+\mu((e_{1,k}^1)^2+(e_{1,k}^2)^2)+\mu((e_{2,k}^1)^2+(e_{2,k}^2)^2+\mu))}\varphi_k(x)\right\| \\
&+\left\|\sum_{k=1}^{\infty}\frac{\mu(e_{1,i}^1e_{1,i}^2+e_{2,i}^1e_{2,i}^2)(g_{1,i}^\delta-g_{1,i})}{(e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1)^2+\mu((e_{2,i}^1)^2+(e_{2,i}^2)^2+(e_{1,i}^1)^2+(e_{1,i}^2)^2)+\mu^2}\varphi_k(x)\right\| \\
&+\left\|\sum_{k=1}^{\infty}\frac{-\mu((e_{1,i}^1)^2+(e_{2,i}^1)^2+\mu)(g_{2,i}^\delta-g_{2,i})}{(e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1)^2+\mu((e_{2,i}^1)^2+(e_{2,i}^2)^2+(e_{1,i}^1)^2+(e_{1,i}^2)^2)+\mu^2}\varphi_k(x)\right\| \\
&+\left\|\sum_{k=1}^{\infty}\frac{\mu(e_{1,i}^1e_{1,i}^2+e_{2,i}^1e_{2,i}^2)g_{1,i}-\mu((e_{1,i}^1)^2+(e_{2,i}^1)^2+\mu)g_{2,i}}{(e_{1,i}^1e_{2,i}^2-e_{1,i}^2e_{2,i}^1)^2+\mu((e_{2,i}^1)^2+(e_{2,i}^2)^2+(e_{1,i}^1)^2+(e_{1,i}^2)^2)+\mu^2}\varphi_k(x)\right\| \\
\le{}& 3\delta+\left\|\sum_{k=1}^{+\infty}\frac{\mu g_{1,k}}{\mu+\frac{\mu((e_{1,k}^1)^2+(e_{2,k}^1)^2)+(e_{1,k}^1e_{2,k}^2-e_{1,k}^2e_{2,k}^1)^2}{\mu+(e_{1,k}^2)^2+(e_{2,k}^2)^2}}\varphi_k(x)\right\|+\left\|\sum_{k=1}^{+\infty}\frac{\mu g_{2,k}}{\mu+\frac{(e_{1,k}^1e_{2,k}^2-e_{1,k}^2e_{2,k}^1)^2}{e_{1,k}^1e_{1,k}^2+e_{2,k}^1e_{2,k}^2}}\varphi_k(x)\right\| \\
&+\left\|\sum_{k=1}^{+\infty}\frac{\mu g_{2,k}}{\mu+\frac{\mu((e_{1,k}^2)^2+(e_{2,k}^2)^2)+(e_{1,k}^1e_{2,k}^2-e_{1,k}^2e_{2,k}^1)^2}{\mu+(e_{1,k}^1)^2+(e_{2,k}^1)^2}}\varphi_k(x)\right\|+\left\|\sum_{k=1}^{+\infty}\frac{\mu g_{1,k}}{\mu+\frac{(e_{1,k}^1e_{2,k}^2-e_{1,k}^2e_{2,k}^1)^2}{e_{1,k}^1e_{1,k}^2+e_{2,k}^1e_{2,k}^2}}\varphi_k(x)\right\| \\
\le{}& 3\delta+\left\|\sum_{k=1}^{+\infty}\frac{\mu g_{1,k}}{\mu+\frac{(e_{1,k}^1e_{2,k}^2-e_{1,k}^2e_{2,k}^1)^2}{(e_{1,k}^2)^2+(e_{2,k}^2)^2}}\varphi_k(x)\right\|+\left\|\sum_{k=1}^{+\infty}\frac{\mu g_{2,k}}{\mu+\frac{(e_{1,k}^1e_{2,k}^2-e_{1,k}^2e_{2,k}^1)^2}{e_{1,k}^1e_{1,k}^2+e_{2,k}^1e_{2,k}^2}}\varphi_k(x)\right\| \\
&+\left\|\sum_{k=1}^{+\infty}\frac{\mu g_{2,k}}{\mu+\frac{(e_{1,k}^1e_{2,k}^2-e_{1,k}^2e_{2,k}^1)^2}{(e_{1,k}^1)^2+(e_{2,k}^1)^2}}\varphi_k(x)\right\|+\left\|\sum_{k=1}^{+\infty}\frac{\mu g_{1,k}}{\mu+\frac{(e_{1,k}^1e_{2,k}^2-e_{1,k}^2e_{2,k}^1)^2}{e_{1,k}^1e_{1,k}^2+e_{2,k}^1e_{2,k}^2}}\varphi_k(x)\right\| \\
\le{}& 3\delta+\left\|\sum_{k=1}^{+\infty}\frac{\mu g_{1,k}}{\mu+\frac{C_{15}}{\lambda_k^2}}\varphi_k(x)\right\|+\left\|\sum_{k=1}^{+\infty}\frac{\mu g_{2,k}}{\mu+\frac{C_{16}}{\lambda_k^2}}\varphi_k(x)\right\|+\left\|\sum_{k=1}^{+\infty}\frac{\mu g_{1,k}}{\mu+\frac{C_{16}}{\lambda_k^2}}\varphi_k(x)\right\|+\left\|\sum_{k=1}^{+\infty}\frac{\mu g_{2,k}}{\mu+\frac{C_{17}}{\lambda_k^2}}\varphi_k(x)\right\| \\
\le{}& 3\delta+\sup_{k\in\mathbb{N}^+}(D(k))\left(\left\|\sum_{k=1}^{+\infty}\lambda_k^{1+\theta}g_{1,k}\varphi_k(x)\right\|\right)+\sup_{k\in\mathbb{N}^+}(F(k))\left(\left\|\sum_{k=1}^{+\infty}\lambda_k^{1+\theta}g_{2,k}\varphi_k(x)\right\|\right) \\
&+\sup_{k\in\mathbb{N}^+}(E(k))\left(\left\|\sum_{k=1}^{+\infty}\lambda_k^{1+\theta}g_{1,k}\varphi_k(x)\right\|+\left\|\sum_{k=1}^{+\infty}\lambda_k^{1+\theta}g_{2,k}\varphi_k(x)\right\|\right),
\end{aligned}
$$

where $D(k)=\frac{\mu\lambda_k^{1-\theta}}{\mu\lambda_k^2+C_{15}}$, $E(k)=\frac{\mu\lambda_k^{1-\theta}}{\mu\lambda_k^2+C_{16}}$, $F(k)=\frac{\mu\lambda_k^{1-\theta}}{\mu\lambda_k^2+C_{17}}$. For constant $p>0$, $\mu>0$, $\beta>0$ and $s\ge\lambda_1>0$, we claim

$$
F(s)=\frac{\mu s^{1-p}}{\beta+\mu s^2}\le\begin{cases} C\mu^{\frac{1+p}{2}}, & 0<p<1, \\ C\mu, & p\ge 1, \end{cases}
$$

hence invoking the Morozov discrepancy principle (4.4) we have

$$\mu \geq \begin{cases} C_{18}\delta^{\frac{2}{1+\theta}}, & 0<\theta<1, \\ C_{19}\delta, & \theta \geq 1. \end{cases}$$

From (C.5) and (4.7), we get

$$\begin{aligned} &\left\| f_\mu^\delta(x) - f(x) \right\| \leq \left\| f_\mu^\delta(x) - f_\mu(x) \right\| + \left\| f_\mu(x) - f(x) \right\| \\ \leq &\frac{\delta}{\sqrt{\mu}} + C\delta^{\frac{\theta}{\theta+1}} \leq \begin{cases} \dfrac{1}{\sqrt{C_{18}}}\delta^{\frac{\theta}{1+\theta}} + C\delta^{\frac{\theta}{1+\theta}}, & 0<\theta<1, \\ \dfrac{1}{\sqrt{C_{19}}}\sqrt{\delta} + C\delta^{\frac{\theta}{1+\theta}}, & \theta \geq 1. \end{cases} \end{aligned}$$

By a similar argument, we have

$$\begin{aligned} &\left\| u_\mu^{0,\delta}(x) - u^0(x) \right\| \leq \left\| u_\mu^{0,\delta}(x) - u_\mu^0(x) \right\| + \left\| u_\mu^0(x) - u^0(x) \right\| \\ \leq &\frac{\delta}{\sqrt{\mu}} + C\delta^{\frac{\theta}{\theta+1}} \leq \begin{cases} \dfrac{1}{\sqrt{C_{18}}}\delta^{\frac{\theta}{1+\theta}} + C\delta^{\frac{\theta}{1+\theta}}, & 0<\theta<1, \\ \dfrac{1}{\sqrt{C_{19}}}\sqrt{\delta} + C\delta^{\frac{\theta}{1+\theta}}, & \theta \geq 1, \end{cases} \end{aligned}$$

completing the proof. □

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